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Determination of Probability of Component or Subsystem Failure

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ABSTRACT

In this paper, we relate the reliability of the system to the reliabilities of the components or subsystems. We discussed the basic concept of system reliability and present a method to determine probabilities of failure of coherent system components under various conditions, especially forcused on probability of component or subsystem failure before system failure.

Several examples illustrate the procedure.

1. Introduction

Most reliability calculations are performed assuming that components and systems are either functioning or failed. This dichotomy is often a resonable assumption, but the assumption is sometimes made simply because there are no applicable results dealing with more complicated state spaces.

In many practical situations, one is interested in computing the probability that a given component or a set of components fails before the system fails. The components or subsystems of a particular system can be ranked according to their probability of failure before system failure or before a periodic inspection.

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Such rank is useful in improving system reliability, detecting failed components, and generating inspection and repair checklists.

One approach to solving this problem is to obtain a joint probability distribution of times to failure for a given set of components and entire system. This paper extends [6] and discussed the method of determing the probability of having a given set of components failed and another set of components working at the time of system failure is based on the notion of boundary probability.

A method of determining the probability, when the reliability structure of the system and the joint distribution of component time-to-failure T_i are known is discussed in [6].

The path and cut set method for determing system realiablity [2] is used. Problems related to the reliability structure of the system are based on [5]. Our method, applied in this paper, is futher extended to the case when certain components of the system are given as either failed or working.

Notation

e_i	i-th component
$\phi(X)$	system structure function
$h(P)$	system reliability
$Ih(e_j)$	reliability importance of component e_j
T	system time-to-failure
T_i	time-to-failure of component e_i
f_i, F_i, R_i	pdf, cdf, success function of T_i
$e_i(t), \bar{e}_i(t)$	events that component e_i is functioning and failed at time t.
$g(t)$	event that system is functioning at time t

2. Basic concepts of system reliability

In this section, we discussed the deterministic aspects of s-coherent structure. We assume that components are s-independent.

Suppose that the state x_i of the e_i is binary random variable with

$$x_i = \begin{cases} 1 & \text{if } e_i \text{ is functioning} \\ 0 & \text{if } e_i \text{ is failed} \end{cases}$$

similarly, the binary variable ϕ indicates the state of the system

$$\phi(X) = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is failed} \end{cases}$$

The series operator replace n components in series with

$$\phi(X) = \prod_{i=1}^{n} x = \min(x_1, \cdots, x_n) \quad (2-1)$$

while the parallel operator replace n components in parallel with

$$\phi(X) = \coprod_{i=1}^{n} x_i = 1 - \prod_{i=1}^{n}(1-x) = \max(x_1, \cdots, x_n) \quad (2-2)$$

where $\phi(X)$ is system structure function and x_i is state of component i ;
$0 \equiv$ failed, $1 \equiv$ function.

And a k-out-of-n structure functions if and only if at least k of the n components function. The structure function is given by

$$\phi(X) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i \geq k. \\ 0 & \text{if } \sum_{i=1}^{n} x_i < k \end{cases} \quad (2-3)$$

The reliability of the system is given by

$$h(P) = P[\phi(X) = 1] = E[\phi(X)] \quad (2-4)$$

When $p_1 = \cdots = p_n = p$, we will use the symbol $h(P)$ as the reliability function of the structure ϕ

Note that for a coherent structure with independent components

$$h(0) = E[\phi(X) \mid p_1 = 0, \cdots, p_n = 0] = \phi(0) = 0$$
$$h(1) = E[\phi(X) \mid p_1 = 1, \cdots, p_n = 1] = \phi(1) = 1 \quad (2-5)$$

An important and interesting general result concerning the shape of the $h(P)$ curve is following.

Theorem 2.1. (Moore and Shannon) Let $h(P_0) = p_0$ for some $0 < p_0 < 1$

$$h(P) < p \quad \text{for} \quad 0 \leq p < p_0$$
$$h(P) > p \quad \text{for} \quad p_0 < p \leq 1$$

This means that for any network the function $h(P)$ crosses the diagonal line of slop 1 at most once in the interval (0, 1).

We shall not present the details of the Moore-Shannon proof, but we shall derive the result for a more general class of networks.

Lemma 2.2 (Proschan F.) The following identity holds for the reliability function :

$$h(P) = p_i h(1_i, P) + (1 - p_i) h(0_i, P) \qquad (2-6)$$

The corresponding monotonicity property for reliability function is given following theorem.

Therom 2.3 Let $h(P)$ be the reliability function of coherent structure.
Then $h(P)$ is strictly increasing in each p_i for $0 < p_i < 1$

proof. from (2-6)

$$\frac{\partial h}{\partial P} = h(1_i, P) - h(0_i, P) \qquad (2-7)$$

so that $$\frac{\partial h}{\partial P} = E[\phi(1_i, X) - \phi(0_i, X)] \qquad (2-8)$$

since ϕ is increasing then

$$\phi(1_i, X) - \phi(0_i, X) \geq 0$$

Thus equation (2-8) is positive, and the desired result follows.

Next, we develop a measure of the reliability importance of each component, which takes into account component reliability as well as system structure. Such a measure can be very useful in system analysis in determining those components on which additional reserch and development effort can be most profitably expended. It would seem reasonable to measure the importance of a component in contributing to system reliability by the rate at which system reliability improves as the reliability of the component improves. From equation (2-6) we present the following definition.

The reliability importance $Ih(e_j)$ of component j is given by

$$Ih(e_j) = \frac{\partial h(P)}{\partial P_j} \qquad (2-9)$$

Example 2.4. The bridge structure is shown in the following diagram

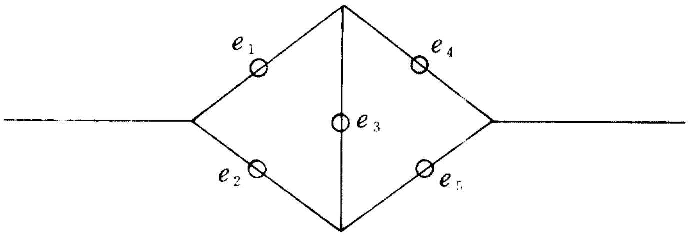

Fig 1. 5-component Bridge system

the minimal path sets are :

$$P_1 = \{e_1, e_4\},\ P_2 = \{e_2, e_5\},\ P_3 = \{e_1, e_3\ e_5\},\ P_4 = \{e_2, e_3\ e_4\}$$

System structure function

$$\begin{aligned}\phi(X) &= x_1x_4 \ \amalg\ x_2x_5 \ \amalg\ x_1x_3x_5 \ \amalg\ x_2x_3x_4 \\ &= [\,1-(1-x_1x_4)(1-x_2x_5)\,] \ \amalg\ [\,1-(1-x_1x_3x_5)(1-x_2x_3x_4)\,] \\ &= (x_1x_4 + x_2x_5 - x_1x_2x_4x_5) \ \amalg\ (x_1x_3x_5 + x_2x_3x_4 - x_1x_2x_3^2x_4) \\ &= 1-[\,1-(x_1x_4+x_2x_5-x_1x_2x_4x_5)\,][\,1-(x_1x_3x_5+x_2x_3x_4-x_1x_2x_3^2x_4)\,]\end{aligned}$$

System reliability

$$\begin{aligned}h(P) = 1 - [\,1 - (p_1p_4 + p_2p_5 - p_1p_2p_4p_5)\,][\,1 - (p_1p_3p_5 + p_2p_3p_4 \\ - p_1p_2p_3^2p_4)\,]\end{aligned}$$

Then, from (2-9)
reliability importance of e_1 is

$$\begin{aligned}Ih(e_1) = \frac{\partial h(P)}{\partial P_1} &= (p_4 - p_2p_4p_5)(1-p_1p_3p_5 + p_2p_3p_4 - p_1p_2p_3^2p_4) \\ &+ (1-p_1p_4-p_2p_5-p_1p_2p_4p_5)(p_3p_5-p_2p_3^2p_4)\end{aligned}$$

Simitarly, we calculate reliability importance of other components.

3. Probability of component or subsystem failure before system failure

Here we compute the probability that component e_i will be failed at the time of system failure $P_r[T_i \leq T]$.

The event $\{T_i < T\}$ encompasses all the cases when e_i faills before system failure

$$P_r[T_i < T] = \int_0^{\infty} P_r\{T_i < T \mid T_i = t\} f_i(t)dt$$

$$= \int_0^{\infty} P_r\{g(t) \mid \bar{e}_i(t)\} f_i(t)dt \qquad (3-1)$$

The event $\{T_i = T\}$ encompasses all the cases when the system functions if component e_i functions and system fails when component e_i fails.

From the definition of the boundary condition for component e_i we have following equation :

$$P_r[T_i = T] = \int_0^{\infty} f_i(t) P(B_i(t))dt \qquad (3-2)$$

where $P[B_i(t)] = P_r\{g(t) \mid e_i(t)\} - P_r\{g(t) \mid \bar{e}_i(t)\}$ (3-3)

from (3-1), (3-3)

$$P_r[T_i \leq T] = P[T_i < T] + P[T_i = T]$$

$$= \int_0^{\infty} P_r\{g(t) \mid e_i(t)\} f_i(t)dt \qquad (3-4)$$

Example 3.1 (1-out-of-n: F) consider an n-component series system shown in figure 2.

Fig 2. n-component series system

$$P\{g(t) \mid e_i(t)\} = \prod_{\substack{j=1 \\ j \neq i}}^{n} R_j(t)$$

Hence

$$P_r[T_i \leq t] = \int_0^{\infty} f_i(t) \prod_{\substack{j=1 \\ j \neq 1}}^{n} R(t)dt$$

Example 3.2 Consider example 2.4
It can be easily verified that for $i=1$
we have :

$$P_r\{g(t) \mid e_i(t)\} = 1 - \{F_2(t)F_3(t)F_5(t) + F_2(t)R_3(t)F_4(t)F_5(t) + R_2(t)F_4(t)F_5(t)\}$$

$$P_r[T_1 \leq T] = \int_0^\infty f_1(t)\{1-(F_5(t)F_2(t)F_3(t)R_4(t) + F_4(t))\}dt$$

Example 3.3 (1-out-of-n:G) consider an n-component parallel system

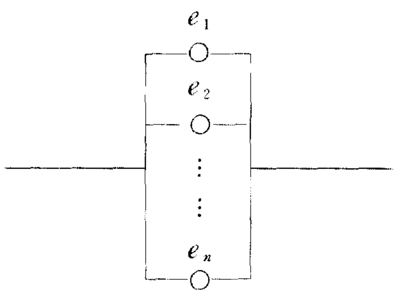

Fig 3. n-component parallel system

since for any i $P_r\{g(t) \mid e_i(t)\} = 1$
we have

$$P[T_i \leq T] = \int_0^\infty f_i(t)dt = 1$$

Now we investigate the probability that a set of component has failed at the time of system failure : whereas another set of component is functioning. That is, we want to determine the probability of a random event of the type

$$E = \{\bigcap_{e_i \in A} T_i < T, \bigcap_{e_j \in B} T_j > T, \bigcap_{e_k \in C} T_k < \infty\}$$

where A, B, C are partitions of the set of all components.(A is the set of components failed before the system failure, and B is the set of components functioning at the time of systim failure.)

$$E_m = \{T_m = T, \bigcap_{e_i \in A_m} T_i < T, \bigcap_{e_j \in B} T_j > T, \bigcap_{e_k \in C} T_k < \infty\}$$

where A_m is the set A without component e_m.

E can be represented as a disjoint union of the following events :

$$E = E' \cup (\bigcup_{e_m \in A} E) \cup E'' \qquad (3-5)$$

where $E'' = \{ \bigcap_{e_i \in A} T_i < T, \bigcap_{e_j \in B} T_j > T, \bigcap_{e_k \in C} T_k < \infty \}$

and E' includes all events from E which do not belong to E_m or E''.

Since E' is a disjoint union of the events where two or more components fail simultaneously at time T.

$$\text{then } P_r [E'] = 0 \qquad (3-6)$$

From (3-5) and (3-6) we have

$$P_r [E] = \sum_{e_m \in A} P_r \{E_m\} + P_r [E''] \qquad (3-7)$$

For given m, $P_r\{E_m\}$ is the conditional boundary probability for component e_m, under the condition that components in the set A_m have failed before system failure occurred and components in set B are still functioning at the time of system failure.

Therefore

$$P_r [E_m] = \int_0^\infty f_m(t) \prod_{e_i \in A_m} F_i(t) \prod_{e_j \in B} R_j(t) P_r [B_m \mid G(t)] dt \qquad (3-8)$$

where $G = A_m \cup B$

The conditional boundary probability $P_r [B_m \mid G(t)]$ is

$$P_r [B_m \mid G(t)] = P_r \{ g(t) \mid e_m(t), \bigcap_{e_i \in A_m} e_i(t), \bigcap_{e_j \in B} e_j(t) \}$$

$$- P_r \{ g(t) \mid \bigcap_{e_i \in A} e_i(t), \bigcap_{e_j \in B} e_j(t) \} \qquad (3-9)$$

$$P_r [E''] = \int_0^\infty \sum_{e_k \in C} f_k(t) \prod_{e_i \in A} F_i(t) \prod_{e_j \in B} R_j(t) P_r [B_k \mid G'(t)] dt \qquad (3-10)$$

where $G' = A \cup B$

$P_r[\,B_k \mid G'(t)\,]$ is computed similarly as in (3−9)
We obtained $P_r\{E\}$ by combining (3−7) − (3−10).

Example 3.4 consider example 3.2

Let $A = \{1, 3\}$, $B = \{4\}$, $C = \{2, 5\}$

The following relations hold :

$$
\begin{aligned}
P_r[\,B_1 \mid G(t)\,] &= P_r\{g(t) \mid e_1(t), \overline{e}_3(t), e_4(t)\} \\
&\quad - P_r\{g(t) \mid \overline{e}_1(t), \overline{e}_3(t), e_4(t)\} \\
&= [\,1 - F_2(t)F_5(t)\,] - R(t) = F_2(t)F_5(t)
\end{aligned}
$$

$$
\begin{aligned}
P_r[\,B_3 \mid G(t)\,] &= P_r\{g(t) \mid e_3(t), \overline{e}_1(t), e_4(t)\} \\
&\quad - P\{g(t) \mid \overline{e}_3(t), e_1(t), e_4(t)\} \\
&= R_2(t) - R_2(t) = 0
\end{aligned}
$$

$$
\begin{aligned}
P_r[\,B_2 \mid G'(t)\,] &= P_r\{g(t) \mid e_2(t), \overline{e}_1(t), \overline{e}_3(t), e_4(t)\} \\
&\quad - P\{g(t) \mid \overline{e}_2(t), \overline{e}_1(t), e_3(t), e_4(t)\} \\
&= 1 - 0 = 1
\end{aligned}
$$

$$
\begin{aligned}
P_r[\,B_5 \mid G'(t)\,] &= P_r\{g(t) \mid e_5(t), \overline{e}_1(t), \overline{e}_3(t), e_4(t)\} \\
&\quad - P_r[\,g(t) \mid \overline{e}_5(t), \overline{e}_1(t), \overline{e}_3(t), e_4(t)\,] \\
&= R_2(t) - R_2(t) = 0
\end{aligned}
$$

Hence

$$P_r[\,E\,] = \int [f_i(t)F_2(t)R_5(t) + f_2(t)F_1(t)]\,F_3(t)R_4(t)dt$$

We hope that our method can be extended to the case of non-s-independent random variables.

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