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The Decision of Bounds for Consecutive k out of n Structure with Sink-Source Pole

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ABSTRACT

The derived expressions and computations of the system reliability in the consecutive k out of n failure structure with sink-source pole are discussed and a decisive lower and upper bounds as well as exact system reliability are presented in this paper. Good bounds of the system reliability can be easily computed and are illustrated in the numerical example.

1. Introduction

Generally, a system with n components in sequence is called consecutive k out of n failure system if the system fails whenever k consecutive components fail [1], [7]. Now care should be taken as to if the source and the sink are also considered components of the systems.

Especially, in the oil or water pipeline system and the telecommunication system, sink-source and the intermediate stations are all the same kind of relayed stations. Thus both source and sink are considered components of the systems.

We are interested in a decisive lower and upper bounds, by the given reliabilities of

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components, as well as exact system reliability in the consecutive k out of n failure structure with sink and source pole. In section 3, we discuss the system reliability of consecutive k out of n failure structure with sink-source pole in terms of component reliability as the traditional generalized medthod for computing. In section 4, we give a decisive bounds on the consecutive k out of n failure system reliability and exact value of system reliability by the new method for computing. . Section 5 is numerical example for bounds and exact system reliability of consecutive k out of n failure system with sink-source pole.

2. Notation

- n : number of components excluding sink-source pole
- k : minimum number of consecutive failed components
- P : component probability, all components have $i.\ i.\ d.$ life distributions
- X_i : state of component $i\,(i = 1, 2, \cdots, n.)$
- $\Phi(x)$: structure function of the system.
- u : random variable indicating the index of first 0 in vector of component states (X).
- v : random variable indicating the index of first 1 after u position in X.
- $B_j(X)$: the indicator function of the jth minimal cut set
- $E(X)$: Expectation of state X
- Pr : Probability
- $ReL(p, k, n)$: System reliability of a consecutive k out of n failure structure
- $ReL^*(p, k, n)$: System reliability of a consecutive k out of n failure structure with sink-source pole

3. Consecutive k out of n Failure System by Minimal Cutset

Let X be a n-vector such that the component i of X is 1 or 0 depending on whether component i of the system is functioning or not, respectively. Thus the vector X indicates which of the components are operating and which have failed.

We further suppose that whether or not the system as a whole is operating is completely determined by the state vector X. Specifically, it is suppose that there exists a function $\Phi(X)$ such that $\Phi(X) = 1$, if the system is operating when the state vector is X. $\Phi(X) = 0$, if the system is failed when the state vector is X. A state vector X is called a cut vector if $\Phi(X) = 0$. If, in addition, $\Phi(X) = 1$, for all $Y > X$, then X is said to be a minimal cut vector. If X is a minimal cut vector, the set $C = \{\, i : X_i = 0 \,\}$ is called a minimal cut set. Let

$C_1, C_2, \cdots, C_k$ denote the minimal cut sets of a given system. We define $B_j(X)$, the indicator function of the jth minimal cut set, by

$$B_j = \begin{cases} 1, & \text{if at least one component of the } j\text{th minimal cut set is functioning} \\ 0, & \text{if all of the components of the } j\text{th minimal cut set are not functioning} \end{cases}$$
$$= \max_{i \in C_j} X_i .$$

It follows that

$$\Phi(X) = \prod_{j=1}^{k} B_j(k)$$
$$= \prod_{i=1}^{k} \max X_i . \qquad (1)$$

Since $B_j(X)$ is a parallel structure function of the components of the jth minimal cut set, $Eq(1)$ represents an arbitrary system as a series arrangement of parallel systems [2] [3] [8].

(1) for $k=1$ in the consecutive k out of n failure system

This case is like as series system, generally, any permutation is invariant optimal. The number of minimal cutset is n, and the size of cutset is just $k=1$.

$$ReL(p, l, n) = P\{\Phi(X)=1\}$$
$$= P\{X_i = 1 \quad \text{for all} \quad i = 1, 2, \cdots, n\}$$
$$= \prod_{i=1}^{n} P_i \qquad (2)$$

(2) $k=k$ in the consecutive k out of n failure system (where $k > 1$).

the number of minimal cutsets are $n-k+1$

$$ReL(p, k, n) = E[\max(1, 2, \cdots, k) \cdot \max(2, 3, \cdots, k+1)$$
$$\cdot \max(3, \cdots, k+1, k+2) \cdots \cdot \max(n-k, n-k+1, \cdots, n-1)$$
$$\cdot \max(n-k+1, n-k+2, \cdots, n)]$$
$$= E[\{1-(1-X_1)\cdots(1-X_k)\}\{1-(1-X_2)\cdots(1-X_k)(1-(1$$
$$-(1-X_{k+1})\}\cdot\cdots\cdot\{1-(1-X_{n-k})(1-X_{n-k+1})\}\cdot\cdots$$
$$\cdot(1-X_{n-1})\}\{1-(1-X_{n-k+1})(1-X_{n-k+2})\cdot\cdots\cdot(1-X_n)\}] \qquad (3)$$

Now let's consider about the consecutive *k* out of *n* failure system including the sink-source pole.

$$ReL^*(p,k,n) = E[\max(X_0 \cdot X_{n+1})] \cdot ReL(p,k,n)$$
$$= E[\{1-(1-X_0 \cdot X_{n+1})\}\{1-(1-X_1)\cdots(1-X_k)\}\{(1-(1-(1-X_2)\cdots(1-X_k)(1-X_{k+1})\}\cdot\cdots\cdot\{1-(1-X_{n-k})(1-X_{n-k+1})\cdot\cdots\cdot(1-X_{n-1})\}\{1-(1-X_{n-k+1})(1-X_{n-k+2})\cdot\cdots\cdot(1-X_n)\}] \quad (4)$$

4. Exact value and Bounds for System Reliability

In order to use the result of equation(4) to compute the reliability of consecutive *k* out of *n* failure system, all possible combinations of sequences of failed components with size loss than *k* must be specified. This procedure becomes too messy and complicate. So we present a comparatively simplified rotative formulation to compute the system reliability. The derivation of the rotative formulation is subject to the examining the first sequence of consecutive $X_i=0$ in the state vector *X*. If the number of consecutive $X_i=0$ in the first sequence is greater than or equal to *k*, then the system is failed. If the number of consecutive $X_i=0$ in the first sequence is less than *k*, then the system reliability is a consecutive *k* out of *n* failure system.

Also if the number of consecutive $X_i=0$ in the first sequence is less than *k*, then the reliability of the system is equal to the reliability of a consecutive *k* out of *n* failure system where *n* is strictly smaller than *n*.

Since the reliability of a consecutive *k* out of *n* failure system for all $n < k$ is exactly 1 by definition [3] [7], we can rotatively compute the reliability of consecutive *k* out of *n* failure system for $n^*-2 \geq k$.

(1) Exact Value System Reliability by the New Ratative Formulation

Now, we can get the new rotative formula which may easily and simply compute the reliability of consecutive *k* out of n^* failure system(with Sink-Source Pole) as following :

$$ReL^*(p,k,n^*) = Pr\{\text{the system is operating}\}$$
$$= \sum_u \sum_v \sum_1 1 \cdot Pr[\Phi(x)=1 \mid U=u, V=v] \cdot Pr[U=u, V=v]$$
$$= \sum_1 \sum_u \sum_v Pr\{\Phi(x)=1 \mid U=u, V=v\}$$

$$= \sum_{1} 1 \cdot Pr\{\Phi(x)=1\}$$

$$= \sum_{u=1}^{n-k+1} \sum_{v=u+1}^{u+k-1} Pr[\Phi(x)=1 \mid U=u, V=v] \cdot Pr[U=u, V=v] + P^{n-k+3}$$

$$= \sum_{u=1}^{n-k+1} \sum_{v=u+1}^{u+k-1} Pr\{\text{the system is operating} \mid U=u, V=v\} \cdot$$

$$(1-P)^{v-u} P^{u+2} + P^{n-k+3} \tag{5}$$

U is the random variable indicating the index of first 0 in vector of component states(X) and V is the random variable indicating the index of first 1 after U position in X in the cut set for the consecutive k out of n failure system. Hwan-oh table for the indexing of U and V shows in [7], and by this indexing table we may have the $Pr\{U=u, V=v\}$ is equal to the $\{P^u(1-p)^{v-u} P^2\}$, and $Pr\{U>n-k+1\} = P^{n-k+1}$

For $u+1 \le v \le u+k-1$, the first sequence of failed components doesn't constitute a cut set. Furthermore, since $X_v=1$, the event that the consecutive k out of n failure system is operating now is equivalent to the event that a consecutive k out of $(n-v)$ failure system is operating.

Hence, from the equation (5), $Pr\{$ the system is operating $U=u, V=v\}$ is reliability is :

$$ReL^*(p,k,n^*) = \sum_{u=1}^{n-k+1} \sum_{v=u+1}^{u+k-1} Rel(p,k,n-v) \cdot p^{u+2} \cdot (1-p)^{u-v} + p^{n-k+3}$$
$$[\text{ if } k>n-v\ge 0,\ Rel(p,k,n-v)=1\ ]$$
$$[\text{ if } n-v<0,\ Rel(p,k,n-v)=0\ ] \tag{6}$$

(2) Lower and Upper Bounds of System Reliability

In many cases, bounds of reliabily would rather be often used instead of exact value of system reliability. In this part we give the cutting lower and upper bounds on system reliability. In the independent k out on n failure system, our approach to obtaining bounds on $Rel(p,k,n)$ is based on expressing the desired probability as the probability of the intersection of events. To do so set, $A_1, A_2, \cdots, A_s$ denote the minimal path sets as before, and define the events D_i

$i=1, 2, \cdots$ by $D_i = \{$ at least one component in A_i has failed $\}$. Now since the system will have failed if and only if at least one component in each of the minimal path sets has failed we have that [1] [2] [8]

$$Pr(D_i \mid D_1 D_2 \cdots D_{i-1}) \ge P(D_i)$$

and so, the unreliability is

$$1 - Rel(p, k, n) = P(D_1 D_2 \cdots D_{i-1}) = Pr(D_1) Pr(D_2 \mid D_1) \cdots Pr(D_s \mid D_1 D_2 \cdots D_{s-1}) \geq \prod_i Pr(D_i)$$

or, equivalently, the upper bound is

$$Rel(p, k, n) \leq 1 - \prod_i (1 - \prod_{j \in A_j} P_j)$$

To obtain a bound in the other direction, let $C_1, \cdots, C_r$ denote the minimal cut sets and define the events $M_1, \cdots, M_n$ by M_i ={ at least one component in C is functioning } . Then, since the system will function if and only if all of the events M_i occur, we have

$$Rel(p, k, n) = P_r(M_1 M_2 \cdots M_r) = P_r(M_1) P_r(M_2 \mid M_1) \cdots P_r(M_r \mid M_1 \cdots M_{r-1}) \geq \prod_i P_r(M_i)$$

Where the last inequality is established in exactly the same manner as for D_i. Hence, the lower bound is

$$Rel(p, k, n) \geq \prod_i [1 - \prod_{j \in C_i} (1 - P_j)] \quad (8)$$

and we thus have the following bounds for the reliability function by the equation (7) and (8)

$$\prod_i [1 - \prod_{j \in C_i} (1 - P_j)] \leq Rel(p, k, n) \leq [1 - \prod_i (1 - \prod_{j \in C_i} P_j) \quad (9)$$

It is to be expected that the upper bound should be close to the actual $Rel(p, k, n)$ if there is not too much overlap in the minimal path sets, and the lower bound to be close if there is not too much overlap in the minimal cut sets [1] [2] [8].

Now, let's deeply consider the consecutive k out of n failure system in the bounds since the consecutive system is clearly different from the independent k out of n failure system like as section 3, any k consecutive components constitute a minimal cut set. That is, there are $n - k + 3$ such minimal cut sets in the consecutive k out of n^* failure system with sink-source pole. If the system is operating, then there is at least one operating component in each cut set. Hence, the lower bound of the consecutive k out of n failure system with sink-source pole is

$$Rel^*(p, k, n) \geq (1 - (1 - p^k))^{n-k+1} \cdot p^2 \quad (10)$$

This above equation (10) is same as the equation (8) multiplied by p^2 in the lower bound of system reliability.

To obtain an upper bound on $Rel^*(p, k, n)$, let's suppose $n = n_1 + n_2 + \cdots + n_i$. Then the n system is operating implies that n_1 system, n_2 system $\cdots$ and n_i system are operating since there doesn't exist any sequence of consecutive k failed components. Thus,

$$Rel^*(p, k, n) \le \{ p^2 \}\{ Pr[n_1 \ \Phi(x)=1] \} \cdots \{ Pr[n_i \ \Phi(x)=1 \}.$$

For this, we need to partition the consecutive k out of n failure system into $[n/k]+3$ subsystem where each subsystem has k consecutive components except the sink-source pole and the last part which has $n-k\beta$ components (where β is an integer value after abandoning the remainder in the value of n/k). Since $n-k$ is greater than or equal to 0 and clearly less than k consecutive components, the last subsystem and sink-source pole could not fail. Hence, the upper bound of system reliability is

$$Rel^*(p, k, n) \le \{ p^2 \}(1-(1-p)^k)^{n/k} \tag{11}$$

The number of components of each subsystem is k execpt the remainder of last subsystem and sink-source pole.

We may feel that the value of consecutive system reliability is obviously different from the value of independent system reliability. This upper bound is cut because the equality holds for the consecutive n out of n by p^2 failure system from the k out of k failure system like as parallel system. Hence, the bound of consecutive k out of n failure system with sink-source pole is

$$p^2\{(1-(1-p)^k)^{n-k+1} \le Rel^*(p, k, n) \le p^2 \cdot \{1-(1-p)^k\}^{n/k} \tag{12}$$

5. Numerical Example

To illustrate and to compare bounds and exact value of system reliability, we consider the consecutive 3 out of 6 failure system with sink-source pole. By the rotative formula, the exact value of system reliability is as following :

$$\begin{aligned} ReL^*(p, 3, 8) &= [ReL(p, 3, 4) \cdot p^3(1-p) + ReL(p, 3, 3) \cdot p^3(1-P)^2] \\ &\quad + [ReL(p, 3, 3) \cdot p^4(1-p) + ReL(p, 3, 2) \cdot p^4(1-P)^2] \\ &\quad + [ReL(p, 3, 2) \cdot p^5(1-p) + ReL(p, 3, 1) \cdot p^5(1-P)^2] \\ &\quad + [ReL(p, 3, 1) \cdot p^6(1-p) + ReL(p, 3, 0) \cdot p^6(1-P)^2] + p^7 \\ &= 6P^4 - 8P^5 + 3P^6 \end{aligned}$$

here, $Rel(p, 3, 2) = Rel(p, 3, 1) = Rel(p, 3, 0) = 1$ and $Rel(p, 3, n<0) = 0$.

The lower bound and upper of the consecutive 3 out of 6 failure system with sink-source pole are :

Lower bound $= p^2(1-(1-p)^3)^4$

and Upper bound $= p^2 \cdot (1-(1-p)^3)^2$

If the 3 out of 6 failure system is not consecutive but independent, the upper bound is p^2 $[1-(1-p)^3]^6$. and the lower bound is $p^2 \cdot [1-(1-p)^3]^{20}$.

The following table show the each value for the exact value and the bound of Rel^* $(p, 3, 8)$

TABLE

Component	Independent $Rel^*(p, 3, 8)$			Consecutive $Rel^*(p, 3, 8)$		
Reliability	Exact value	Lower	Upper	Exact value	Lower	Upper
0. 50	0. 0391	0. 0173	0. 0802	0. 1719	0. 1465	0. 1914
0. 60	0. 0952	0. 0689	0. 2035	0. 2955	0. 2763	0. 3154
0. 70	0. 3047	0. 2834	0. 3957	0. 4489	0. 4392	0. 4639
0. 80	0. 5767	0. 5450	0. 6129	0. 6226	0. 6198	0. 6298
0. 90	0. 7971	0. 7940	0. 8086	0. 8070	0. 8068	0. 8084
1. 00	1	1	1	1	1	1

6. Conclusion

This paper has presented a new simplified formulation to compute the exact value of the system reliability of consecutive k out of n failure structure with sink-source pole by using the rotative formulation in terms of Hwan Oh Table [7]. This new formulation has showed so scientific technique as comparing with traditional computation as applying the minimal cut set.

In the bounds of system reliability, Good bound of system reliability sometimes is needed. This paper also gave the lower and upper bounds of consecutive k out of n failure system reliability including sink-source pole. Especially, to obtain a bound, we generally partitioned the consecutive k out of n failure system into subsystem which has k consecutive components except the last part having $n-k\beta$ ($0 \leq n-k\beta < k$) and sink-source pole part.

It is hoped that the new formulation for exact system and bound reliability in this paper would be useful for the real world and further reseach is being pursued in system reliability.

REFERENCES

1. Ang, A. H. S., Ma, H. F. (1981), "*On the Reliability of Structural Systems,*" Proc. 3rd International Conf. Structural Safety and Reliability, Elsevier Scientific Pubokshing, pp. 295−314.
2. Barlow, R. E. and Proschan, F. (1975), "*Statistical Theory of Reliability and Life Testing : Probability Models*", Holt, Rinchart, and Winston, New York.
3. Barlow, R. E., Heidtmann, K. D., (1984), "*Computing k out of n system reliability*", IEEE Trans. Rel, Vol. R−33, pp. 322−323.
4. Collins, J. A. (1981), "*Failure of Materials in Mechanical Design,*" Wiley-Interscience.
5. Martz, H. F., Waller, R. A. (1982), "*Bayesian Reliability Analysis*", John Wiley and Sons.
6. Jain, S. P., Gopal, K. (1985), "*Recursive Algorithm for Reliability Evaluation of k out of n : G system*", IEEE Trans Rel, Vol. R−34, pp. 144−146. Approach McGraw Hill Co.
7. Oh, C. H., (1993), "*A New Formulation of System Reliability for Consecutive k out of n Structure with Sink-Source Pole*", Journal of the KSQC., Vol 21, No. 1, pp. 121−135.
8. Ross, S. M. (1980), "*Probability and Mathematical Statistics*", Academic Press. Inc.