Kyungpook Mathematical Journal
Vol.33, No.2, 249-253, December 1993

PERFECTLY NORMALLY PREORDERED SPACES

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It is well known that a topological space $(X, \mathcal{T})$ is perfectly normal *iff* for any nonempty closed set A in the space and any point $b \notin A$, there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = A, f(b) = 1$ [7]. In this paper, we will show that the similar result is obtained by defining analogously the perfectly normally preordered space as the perfectly normal space.

First we fix some notatioins and terminologies. Let $(X, \leq)$ be a preordered set. For a subset A of X, we write $d(A) = \{y \in X \mid y \leq x \text{ for some } x \in A\}$ and $i(A) = \{y \in X \mid x \leq y \text{ for some } x \in A\}$. A subset A of X is decreasing (increasing, respectively) if $A = d(A)$ ($A = i(A)$, respectively). We say that a function f from X to a preordered space Y is increasing (decreasing, respectively) if $x \leq y$ implies $f(x) \leq f(y)$ ($f(y) \leq f(x)$, respectively) in Y.

In a preordered topological space $(X, \mathcal{T}, \leq)$, for a subset A of X, $D(A)$ denotes the smallest closed decreasing subset containing A. Dually $I(A)$ denotes the smallest closed increasing subset containing A.

Remark. In a preordered space $(X, \leq)$, if A and B are the decreasing (increasing, respectively) subsets of X, then $A \cap B$ is also a decreasing (increasing, respectively) subset of X.

Perfectly normally preordered space

The preordered topological space $(X, \mathcal{T}, \leq)$ is said to be *perfectly normally preordered* if the space is normally preordered and every decreasing

Received May 31, 1993
This paper was supported by the Ministry of Education, 1992.

(increasing, respectively) closed subset in the space is a decreasing (increasing, respectively) G_δ-set, that is, $\cap_{i=1}^{\infty} G_i$, where G_i is a decreasing (increasing, respectively) open set. The following lemma can be found in [5].

LEMMA. *A preordered topological space $(X, \mathcal{T}, \leq)$ is normally preordered iff given any decreasing (increasing, respectively) closed subset A of X and any decreasing (increasing respectively) open subset of V of X with $A \subset V$, there exists a decreasing (increasing, respcetively) open subset W of X such that $A \subset W$, $D(W) \subset V$ ($I(W) \subset V$, respectively).*

We obtain the following result.

THEOREM. *A preordered topological space $(X, \mathcal{T}, \leq)$ is perfectly normally preordered iff for any nonempty decreasing closed set A in X and $b \in X$ with $b \notin A$, there exists an increasing continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = A$ $f(b) = 1$, and dually.*

Proof. Suppose that such functions exist. Let A be a decreasing closed subset and let B be an increasing closed subset of X such that $A \cap B = \phi$. Pick $a \in A, b \in B$. Then there are a continuous increasing function $f : X \to [0,1]$ such that $f^{-1}(0) = A, f(b) = 1$ and a continuous decreasing function $g : X \to [0,1]$ such that $g^{-1}(0) = B, g(a) = 1$. Hence $\psi = f - g$ is a continuous increasing function from X to $[-1,1]$, and $\psi < 0$ for all $x \in A$, $\psi > 0$ for all $x \in B$. The open set $\psi^{-1}(y < 0)$, $\psi^{-1}(y > 0)$ are disjoint and $A \subset \psi^{-1}(y < 0)$, $B \subset \psi^{-1}(y > 0)$. Moreover $\psi^{-1}(y < 0)$ is decreasing. For, suppose that $k \leq l$ and $l \in \psi^{-1}(y < 0)$; then $\psi(k) \leq \psi(l)$ and $\psi(l) < 0$; hence $\psi(k) < 0$ and consequently $k \in \psi^{-1}(y < 0)$. And dually one can easily show that $\psi^{-1}(y > 0)$ is increasing. Hence $(X, \mathcal{T}, \leq)$ is normally preordered. Now, let A(B, respectively) be any decreasing (increasing, respectively) closed subset of X. Then there are the following two cases (i) or (ii).

(i) $A = X$ or $A = \phi$ ($B = X$ or $B = \phi$, respectively). In this case, it is a G_δ-set obviously.

(ii) $A \neq X$ and $A \neq \phi$ ($B \neq X$ and $B \neq \phi$, respectively). Consider the case that A is a decleasing closed subset of X. There exists $b \notin A$. Then there exists a continuous increasing function f from X to $[0,1]$ such that $f^{-1}(0) = A$, $f(b) = 1$. It follows that $A = \cap_{n=1}^{\infty} f^{-1}(y < \frac{1}{n})$ is a decreasing G_δ-set. For, suppose that $k \leq l$ and $l \in f^{-1}(y < \frac{1}{n})$; then $f(k) \leq f(l)$ since f is an increasing function, and $f(l) < \frac{1}{n}$; hence $f(k) < \frac{1}{n}$ and thus $k \in f^{-1}(y < \frac{1}{n})$; therefore $f^{-1}(y < \frac{1}{n})$ is a decreasing open set.

Now consider the case that B is an increasing closet subset of X. There exists $a \notin B$. Then there exists a continuous decreasing function g from X to $[-1,0]$ such that $g^{-1}(-1) = B, g(a) = 0$. It follows that $B = \bigcap_{n=1}^{\infty} g^{-1}(y < -\frac{n}{n+1})$ is an increasing G_δ-set. For, suppose that $k \leq l$ and $k \in g^{-1}(y < -\frac{n}{n+1})$; then $g(l) \leq g(k)$ since g is a decreasing function, and $g(k) < -\frac{n}{n+1}$; hence $g(l) < -\frac{n}{n+1}$ and $l \in g^{-1}(y < -\frac{n}{n+1})$; therefore $g^{-1}(y < -\frac{n}{n+1})$ is an increasing open set. Hence $(X, \mathcal{T}, \leq)$ is perfectly normally preordered.

Now we assume that $(X, \mathcal{T}, \leq)$ is perfectly normally preordered. Let A be a nonempty proper decreasing closed subset of X and let $b \notin A$. Then by hypothesis A is a decreasing G_δ-set, that is, $A = \bigcap_{i=1}^{\infty} G_i$ where G_i is a decreasing open set. Hence there exists n_o such that $b \notin G_{n_0}$. Set $H_1 = \bigcap_{i=1}^{n_0} G_i$. Then $A \subset H_1$ and H_1 is a decreasing open set by REMARK.

Since $(X, \mathcal{T}, \leq)$ is normally preordered, by LEMMA, there exists a decreasing open subset H_1^* such that $A \subset H_1^* \subset D(H_1^*) \subset H_1$. Set $H_2 = G_{n_0+1} \cap H_1^*$. Then H_2 is a decreasing open set and $A \subset H_2, D(H_2) \subset H_1$. Assume that H_1 to H_k, have been determined so that $A \subset H_k$, $D(H_k) \subset H_{k-1}$ and $H_k \subset G_{n_0+k-1}$. We set $H_{k+1} = G_{n_0+k} \cap H_h^*$ where the open decreasing H_k^* is such that $A \subset H_k^* \subset D(H_k^*) \subset H_k$.

Inductively, we construct a sequence $\{H_n\}$ of decreasing open sets such that $A = \bigcap_{n=1}^{\infty} H_n$ since $A \subset H_n \subset G_{n_0+n-1}$, and which satisfies $D(H_{n+1}) \subset H_n \subset X - \{b\}$. Now we take $V(\frac{1}{2^n})$ to be H_{n+1}. Note that $H_1 = V(1), H_2 = V(\frac{1}{2}), H_3 = V(\frac{1}{2^2})$. And we construct a sequence $\{V(\frac{k}{2^n})\}$ of decreasing open sets as follows; Let $q = \frac{k}{2^n}$ such that $0 \leq q \leq 1$, $0 \leq k \leq 2^n$ and $n \geq 0$. With each q we will associate an open decreasing subset $V(q)$ of X such that if $q < q'$ then $D(V(q)) \subset V(q')$. Since $D(H_2) \subset H_1$, $D(V(\frac{1}{2})) \subset V(1)$. Using LEMMA, we can find a decreasing open set, which we let $V(\frac{3}{2^2})$ be such that

$$D\left(V\left(\frac{1}{2}\right)\right) \subset V\left(\frac{3}{2^2}\right),\ D\left(V\left(\frac{3}{2^2}\right)\right) \subset V(1).$$

Since $D(H_3) \subset H_2$, $D(V(\frac{1}{2^2})) \subset V(\frac{1}{2})$. Assume we have defined $V(q)$ for n. We now define $V(q)$ for $n+1$ (and thus for $k = 1, 3, \cdot, 2^{n+1}-1$). Note that the definition of $V(q)$ needs to be given only for odd k: for if k were even, the numerator and denumerator of q could be divided by 2. Because the $V(q)$ have already been constructed for $q = \frac{k}{2^n}$, k odd, we have

$$D\left(V\left(\frac{k-1}{2^{n+1}}\right)\right) \subset V\left(\frac{k+1}{2^{n+1}}\right).$$

[since k is odd, $(D(V(\frac{k-1}{2^{n+1}}))) = D(V(\frac{\frac{k-1}{2}}{2^n}))$, which has already been defined]. Therefore by LEMMA, there exists a decreasing open set, which we let be $V(\frac{k}{2^{n+1}})$ such that

$$D\left(V\left(\frac{k-1}{2^{n+1}}\right)\right) \subset V\left(\frac{k}{2^{n+1}}\right) \subset D\left(V\left(\frac{k}{2^{n+1}}\right)\right) \subset V\left(\frac{k+1}{2^{n+1}}\right).$$

We thus have an inductive definition of $V(q)$ for each q as described. By construction, the collection of $V(q)$ have the prescribed property that if $q < q'$ then $D(V(q)) \subset V(q')$. We define the function f as follows:

$$f(x) = \text{ greatest lower bound } \left\{q|q = \frac{k}{2^n} \text{ and } x \in V(q)\right\} \text{ for } x \in H_1$$

and $f(x) = 1$ for $x \in X - H_1$. Then f is an increasing function. For, if $x \leq y$ and $y \in V(q)$, then $x \in V(q)$ because $V(q)$ is a decreasing subset of X; hence $f(x) \leq f(y)$. And f is a continous function from X to $[0,1]$. For, suppose that $f(x_0) = y_0$ and assume that y_0 is neither 0 nor 1. Let $(y_0 - p, y_0 + p)$ is any neighborhood of y_0 where p is a positive number.Then there are rationals q and q' of the form $\frac{k}{2^n}$ such that $y_0 \in (q, q') \subset (y_0 - p, y_0 + p)$. Then

$V = V(q') - D(V(q))$ is a neighborhood of x_0 and $f(V) \subset (y_0 - p, y_0 + p)$.

If y_0 is either 0 or 1, then the corresponding neighborhoods of 0 and 1, respectively, are $[0, q)$ and $(q', 1]$; but the argument is the same. Hence f is continous. Furthermore, we have $f(b) = 1$ and

$$A \subset f^{-1}(0) = \cap_{n=1}^{\infty} \left\{f^{-1}\left(y < \frac{1}{2^n}\right)\right\} \subset \cap_{n=1}^{\infty}\{H_n\} = A, \text{ that is, } f^{-1}(0) = A.$$

Now, let B be a nonempty proper increasing closed subset of X. Then the same argument can be applied dually so that for B and $a \notin B$, there exists a decreasing continuous function g such that $g^{-1}(0) = B$ and $g(a) = 1$. The proof is complete.

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