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ON g-CLOSED SETS AND g-CONTINUOUS MAPPINGS

Miguel Caldas Cueva

Introduction. The concept of generalized closed set of a topological space (breifly g-closed) was introduced by N. Levine ([8] in 1970). These sets where also considered by W. Dunhan ([5] in 1982) and by W. Dunham and N. Levine ([4] in 1980). Recently, K. Balachandran, P. Sundara and H. Maki ([1] in 1991), defined a new class of mappings called generalized continuous mappings (briefly g-continuous) which contains the class of continuous mappings. The present note has as purpose generalize and improve Theorem 6.3 of N. Levine [8] and to investigate some properties of g-closed sets and g-continuous mappings. Throughout the present note X and Y will always denote topological spaces on which no separation axioms are assumed unless explicity stated. By $f : X \to Y$ we denote a mapping not necessarily continuous f of a topological space X into a topological space Y. When A is a subset of a topological space, we denote the closure of A and the interior of A of $Cl(A)$ and $Int(A)$, respectively.

Definition 1. A subset B of a topological space X is called g-closed in X[8] if $Cl(B) \subset G$ whenever $B \subset G$ and G is open in X. A subset A is called g-open in X if its complement, $X - A$ is g-closed.

Definition 2. A mapping $f : X \to Y$ from a topological space X into a topological space Y is called g-continuous [1] if the inverse image of every closed set in Y is g-closed in X. Clearly it is proved that a mapping $f : X \to Y$ is a g-continuous if and only if the inverse image of every open set in Y is g-open in X.

Definition 3. The intersection of all g-closed sets containing as set A is called the g-closure of A[5] and is denoted by $Cl^*(A)$. This is, for any

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$A \subset X$, $Cl^*(A) = \cap\{F : A \subset F \in \Gamma\}$ where $\Gamma = \{F : F \subset X$ and F is g-closed $\}$

$$\text{If } A \subset X, \text{ then } A \subseteq Cl^*(A) \subseteq Cl(A).$$

For the rest of this article we shall assume that $Cl^*(A)$ is g-closed.

Remark 4. A set B is g-closed if and only if $Cl^*(B) = B$.

Definition 5. Let p be a point of X and N be a subset of X. N is called a g-neighborhood of p in X if there exists a g-open set O of X such that $p \in O \subset N$.

Lemma 6. *Let A be a subset of X. Then, $p \in Cl^*(A)$ if and only if for any g-neighborhood N_p of p in X, $A \cap N_p \neq \emptyset$.*

Proof. Necessity. Suppose that $p \in Cl^*(A)$. If there exists a g-neighborhood N of the point p in X such that $N \cap A = \emptyset$, then by Definition 5, there exists a g-open set O_p such that $p \in O_p \subset N$. Therefore, we have $O_p \cap A = \emptyset$, so that $A \subset X - O_p$. Since $X - O_p$ is g-closed (Definition 1), then $Cl^*(A) \subset X - O_p$. As $p \notin Cl^*(A)$ which is contrary to the hypothesis. If $p \notin Cl^*(A)$, then by Definition 3 there exists a g-closed set F of X such that $A \subset F$ and $p \notin F$. Therefore, we have $p \in X - F$ such that $X - F$ is a g-open set. Hence $X - F$ is a g-neighborhood of p in X, but $(X - F) \cap A = \emptyset$. This is contrary to the hypothesis.

Reasoning as in [10] we obtain:

Definition 7. Let A be a subset of X. A mapping $r : X \to A$ is called a g continuous retraction if r is g-continuous and the restriction r/A is the identity mapping on A.

Theorem 8. *Let A be a subset of X and $r : X \to A$ be a g-continuous retraction. If X is Hausdorff, then A is a g-closed set of X.*

Proof. Suppose that A is not g-closed. Then by Remark 4, there exists a point x in X such that $x \in Cl^*(A)$ but $x \notin A$. It follows that $r(x) \neq x$ because r is g-continuous retraction. Since X is Hausdorff, there exists disjoint open sets U and V in X such that $x \in U$ and $r(x) \in V$. Now let W be an arbitrary g-neighborhood of x. Then $W \cap U$ is a g-neighborhood of x. Since $x \in Cl^*(A)$, by Lemma 6, we have $(W \cap U) \cap A \neq \emptyset$. Therefore, there exists a point y in $W \cap U \cap A$. Since $y \in A$, we have $r(y) = y \in U$ and hence $r(y) \notin V$. This implies that $r(W) \not\subset V$ because $y \in W$. This is contrary to the g-continuity of r. Consequently, A is a g-closed set of X.

Theorem 9. *Let $\{X_i / i \in I\}$ be any family of topological spaces. If*

$f : X \to \Pi X_i$ is a g-continuous mapping, then $Pr_i \circ f : X \to X_i$ is g-continuous for each $i \in I$, where Pr_i is the projection of ΠX_j onto X_i.

Proof. We shall consider a fixed $i \in I$. Suppose U_i is an arbitrary open set in X_i. Then $Pr_i^{-1}(U_i)$ is open in ΠX_i. Since f is g-continuous, we have by Definition 2, $f^{-1}(Pr_i^{-1}(U_i)) = (Pr_i \circ f)^{-1}(U_i)$ g-open in X. Therefore $Pr_i \circ f$ is g-continuous.

N. Levine in ([8], Theorem 6.3) showed that if $f : X \to Y$ is continuous and closed and if B is a g-open (or g-closed) subset of Y, then $f^{-1}(B)$ is g-open (or g-closed) in X. The following theorem is a sligh improvement of this Theorem.

Theorem 10. *If $f : X \to Y$ is g-continuous and closed and if G is a g-open (or g-closed) subset of Y, then $f^{-1}(G)$ is g-open (or g-closed) in X.*

Proof. Let G be a g-open set in Y. Let $F \subset f^{-1}(G)$ where F is closed in X. Therefore $f(F) \subset G$ holds. By Theorem 4.2 of [8] and since $f(F)$ is closed and G is g-open in Y, $f(F) \subset Int(G)$ holds. Hence $F \subset f^{-1}(Int(G))$. Since f is g-continuous and $Int(G)$ is open in Y,

$$F \subset InT(f^{-1}(Int(G))) \subset Int(f^{-1}(G)).$$

Therefore by Theorem 4.2 of [8], $f^{-1}(G)$ is g-open in X. By taking complements, we can show that if G is g-closed in Y, then $f^{-1}(G)$ is g-closed in X.

Under g-continuous open mappings, the inverse image of g-open (or g-closed) sets need not however, be g-open (or g-closed).

Example 11. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = f(c) = a$ and $f(b) = b$. Then f is g-continuous and open. However $\{a, c\}$ is g-closed in Y but $f^{-1}(\{a, c\})$ is not g-closed in X.

By Remark 2 of [1], the composition mapping of two g-continuous mappings is not always g-continuous. However, we obtain the following corollary as an immediate consequence of Theorem 10.

Corollary 12. *Let X, Y and Z be three topological spaces. If $f : X \to Y$ is an closed and g-continuous mapping and $g : Y \to Z$ is a g-continuous mapping, then $g \circ f : X \to Z$ is g-continuous.*

We can conclude our work by generalysing the following well known Theorem which may be found in [3, Theorem 1.5 pg.140].

Theorem A. *Let X be an arbitrary topological space, Y be a Hausdorff space, and $f, g : X \to Y$ be continuous. Then $\{x : f(x) = g(x)\}$ is closed in X.*

Before we prove the generalization of Theorem A, we will need some preliminary results, analogous to that obtained by D. Jankovic [6].

Theorem 13. *If $f : X \to Y$ is g-continuous, with Y Hausdorff space, and F a closed subset of $X \times Y$, then $Pr_1(F \cap G(f))$ is g-closed in X, where $G(f) = \{(x, f(x)) : x \in X\}$ denote the graph of f and $Pr_1 : X \times Y \to X$ represents the projection.*

Proof. Let $x \in Cl^*(Pr_1(F \cap G(f))$, where F is a closed subset of $X \times Y$, let O be an arbitrary open set containing x in X, and let V be an arbitrary open set containing $f(x)$ in Y. Since f is g-continuous, $f^{-1}(V)$ is g-open in X. By Corollary 2.7 of [8] $O \cap f^{-1}(V)$ is g-open in X. By Lemma 6 and since $x \in Cl^*(Pr_1(F \cap G(f)))$,

$$(O \cap f^{-1}(V)) \cap Pr_1(F \cap G(f)) \neq \emptyset.$$

Let $p \in (O \cap f^{-1}(V)) \cap Pr_1(F \cap G(f))$. This implies that $(p, f(p)) \in F$ and that $f(p) \in V$. Therefore $(O_x V) \cap F \neq \emptyset$ and consequently, $(x, f(x)) \in Cl^*(F)$. Since F is closed $(x, f(x)) \in F \cap G(f)$. Hence $x \in Pr_1(F \cap G(f))$ and the result follows of Remark 4.

Corollary 14. *If $f : X \to Y$ has a closed graph and $g : X \to Y$ is g-continuous, then $\{x \in X : f(x) = g(x)\}$ is g-closed.*

Proof. Since $\{x \in X : f(x) = g(x)\} = Pr_1(G(f) \cap G(g))$ and since $G(f)$ is a closed subset of $X \times Y$. The result follows from Theorem 13.

A mapping $f : X \to Y$ is said to be weakly-continuous [7] if for every $x \in X$ and every open set V containing $f(x)$ in Y there exists an open set U containing x in X such that $f(U) \subseteq Cl(V)$. Since a weakly-continuous mapping into a Hausdorff space has a closed graph ([9] Theorem 10), we have by Corollary 14.

Corollary 15. *If $f : X \to Y$ is weakly-continuous, $g : X \to Y$ is g-continuous, and Y is Hausdorff, then $\{x \in X : f(x) = g(x)\}$ is g-closed.*

Finally, the fact that continuity implies weak-continuity gives that Theorem A is a consequence of Corollary 15, if X is $T_{1/2}$ space ([8]: A topological space is $T_{1/2}$ iff every g-closed set is closed).

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Departamento de Matematica Aplicada-IMUFF, Universidade Federal Fluminense Rua São Paulo S/N, CEP: 24210, Niteroi, RJ-Brasil.