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THE SPECTRUM OF LAPLACIAN FOR $SU(4)/SU(2) \otimes SU(2)$ AND ITS APPLICATION

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In this paper, we compute the spectrum of the Laplacian for normal homogeneous manifold $SU(4)/SU(2) \otimes SU(2)$, and, as a by-product of spectrum calculation, prove that the identity map of this manifold is stable as a harmonic map.

1. Introduction and statement of the main results

1.1. Let (M,g) be an n-dimensional compact Riemannian manifold. We denote by Δ the Laplace-Beltrami operator acting on the space $C^\infty(M)$ of all complex valued smooth functions on M, that is,

$$\Delta = -\sum_{i,j} \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} (\sqrt{G} g^{ij} \frac{\partial}{\partial x^j}), \tag{1.1}$$

where the g_{ij} are the components of g with respect to a local coordinate $(x_1, x_2, \cdots, x_n)$, (g^{ij}) is the inverse matrix of (g_{ij}) and $G = \det(g_{ij})$. Then, the spectrum $Spec(M,g)$ of the Laplacian Δ, i.e, the set of all eigenvalues of the Laplacian, consists of

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \to +\infty.$$

Problem. Given a Riemannian manifold (M,g), calculate its spectrum $Spec(M,g)$.

This task seems to be impossible, in general, for nonhomogeneous Riemannian manifolds. For a few Riemannian manifolds, e.g., flat tori, lens spases and symmetric spaces, spectra have been calculated ([8], [9], [11]).

In this paper, we treat a normal homogeneous manifold $(M,g) = SU(4) \; / \; SU(2) \otimes SU(2)$. That is, let $(\cdot,\cdot)$ be an $\mathrm{Ad}(SU(4))$-invariant

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inner product on the Lie algebra $\mathbf{su}(4)$. Let $\mathbf{m}$ be the orthogonal complement to the subalgebra $\mathbf{su}(2) \otimes I_2 + I_2 \otimes \mathbf{su}(2)$ of $SU(2) \otimes SU(2)$ in $\mathbf{su}(4)$ relative to $(\cdot,\cdot)$, so that $\mathbf{su}(4) = \mathbf{su}(2) \otimes I_2 + I_2 \otimes \mathbf{su}(2) + \mathbf{m}$ and $\mathrm{Ad}(SU(2) \otimes SU(2))\ (\mathbf{m}) = \mathbf{m}$, where

$$a \otimes b := \begin{pmatrix} a_{11}b & a_{12}b \\ a_{21}b & a_{22}b \end{pmatrix},$$

$$(a = (a_{ij}), b = (b_{ij}) \in M_2(C)).$$

The tangent space $T_o(SU(4)/SU(2) \otimes SU(2))$ at the origin $o{:=}SU(2) \otimes SU(2)$ can be identified with the subspace $\mathbf{m}$ by

$$\mathbf{m} \ni X \to X_o \in T_o(SU(4)/SU(2) \otimes SU(2)),$$

where $X_o\, f := d/dt\ f(\exp tX\cdot o)|_{t=0}$ for a C^∞-function f on $SU(4)/SU(2)\otimes SU(2)$. An inner product g_o on the tangent space at o defined by $g_o(X_o, Y_o) = (X,Y)$, $X, Y \in \mathbf{m}$, can be uniquely extended to a $SU(4)$-invariant Riemannian metric g on $SU(4)/SU(2) \otimes SU(2)$.

1.2. A harmonic map is a differential map between two Riemannian manifolds which is a critical point of the energy functional. As a variational problem, it is natural to study the stability of a given harmonic map. The identity map id_M of a given Riemannian manifold (M, g) is a harmonic map. (M, g) is said to be stable when the second variation of the energy functional at id_M is non-negative and in the other cases, (M, g) is said to be unstable.

1.3. Under the above preparations **1.1** and **1.2**, we can state the following results.

Theorem. *Let (M, g) be a normal homogeneous Riemannian manifold $(SU(4)/(SU(2) \otimes SU(2)),\ g)$ with the normal metric g which is canonically induced from the Killing form B on the Lie algebra* $\mathbf{su}(4)$ *of $SU(4)$. Then (M, g) is an irreducible symmetric Riemannian manifold, and the least positive eigenvalue of the Laplacian Δ for $(M,\ g)$ is $\frac{9}{8}$.*

Corollary. *$(M,\ g)$ is stable.*

2. Proof of the main results

2.1. In this part, we present some results on the sectra for normal homogeneous Riemannian manifolds, and introduce some Theorems to prove the main results.

The spectrum $Spec(G/K, g)$ of the Laplacian for a normal homogeneous Riemannian manifold G/K can be obtained as follows [9, pp.979-980]. Let $\mathbf{t}$ be a maximal abelian subalgebra of the Lie algebra $\mathbf{g}$ of G. Since the weight of a finite unitary representation of G relative to $\mathbf{t}$ has its value in purely imaginary numbers on $\mathbf{t}$, we consider the weight as an element of $\sqrt{-1}\mathbf{t}^*$, where $\mathbf{t}^*$ denotes the real dual space of $\mathbf{t}$. From the $\mathrm{Ad}(G)$-invariant inner product on $\mathbf{g}$, a positive inner product on $\sqrt{-1}\mathbf{t}^*$ is defined in the usual way and denoted by the same symbol $(\cdot,\cdot)$. Fixing a lexicographic order $>$ on $\sqrt{-1}\mathbf{t}^*$, let P be the set of all positive roots of the complexification $\mathbf{g}^c$ of $\mathbf{g}$ relative to $\mathbf{t}^c$. We denote by δ half the sum of all elements in P. Let $\Gamma(G) = \{H \in \mathbf{t}; \exp H = e\}$ and $I = \{\lambda \in \sqrt{-1}\mathbf{t}^*; \lambda(H) \in 2\sqrt{-1}Z$ for all $H \in \Gamma(G)\}$. An element in I is called a $G-integral\ form$. The elements of

$$D(G) = \{\lambda \in I;\ (\lambda, \alpha) \geq 0 \text{ for all } \alpha \in P\}$$

are called *dominant* G-integral forms. Then there exists a natural bijection from $D(G)$ onto the set $\mathbf{D}(G)$ of all nonequivalent finite dimensional irreducible unitary representation of G which map a dominant G-integral form $\lambda \in D(G)$ to an irreducible unitary representation (V_λ, π_λ) having highest weight λ. For $\lambda \in D(G)$, put $d(\lambda)$ the dimension of the representation V_λ. $d(\lambda)$ is given by

$$d(\lambda) = \Pi_{\alpha \in P}(\lambda + \delta, \alpha)/(\delta, \alpha).$$

A representation (V_λ, π_λ) in $\mathbf{D}(G)$ is called *spherical* relative to K if there exists a nonzero vector $v \in V_\lambda$ such that $\pi_\lambda(k)\ v = v$ for all $k \in K$. Let $\mathbf{D}(G, K)$ be the set of all spherical representations in $\mathbf{D}(G)$ relative to K and $D(G, K) = \{\lambda \in D(G);\ (V_\lambda, \pi_\lambda) \in \mathbf{D}(G, K)\}$. Then the following Theorem is well known.

Theorem 1 [8, Propo. 2.1, p. 558]. *The spectrum $Spec(G/K, g)$ of the Laplacian on the normal homogeneous space $(G/K, g)$ is given by eigenvalues: $(\lambda + 2\delta, \lambda)$, $\lambda \in D(G, K)$.*

Furthermore, we need the following Theorems to prove the main results.

Theorem 2 [1, Th. 7.73, p. 194]. *The Ricci curvature tensor R of a Riemannian symmetric space G/K satisfies*

$$R = \frac{-1}{2} B_{\mathbf{m}},$$

where B is the Killing form of $\mathbf{g}$, $\mathbf{m}$ *is the canonical complementary space of Riemannian symmetric pair* (G, K), *and* $B_{\mathbf{m}}$ *is the restriction of* B *to* $\mathbf{m} \times \mathbf{m}$.

Theorem 3 [10, Th. 1.28, p. 198]. *If a compact Riemannian manifold* (M, g) *is Einstein, i.e., the Ricci tensor* R *satisfies* $R = cg$, *then the identity map* id_M *of* (M, g) *is stable if and only if the first eigenvalues* $\lambda_1(M, g)$ *of the Laplace-Beltrami operator of* (M, g) *acting on* $C^\infty(M)$ *satisfies* $\lambda_1(M, g) \geq 2c$.

Theorem 4 [3, Th. 10, p. 87]. *Let* $(G/K, g)$ *be a symmetric Riemmannian manifold. Then, if* G *is simple,* G/H *is irreducible.*

2.2. The inclusion of $SU(2) \otimes SU(2)$ into $SU(4)$ is the tensor product of two usual linear representations of $SU(2)$. In this section, we use the following notations.

$G = SU(4)$, $G_{(2)} = SU(2)$, $H = (SU(2) \otimes SU(2))$, $M = G/H$,
$T = \{diag[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]; \epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1, |\epsilon_i| = 1, \epsilon_i \in C\}$,
$T_{(2)} = \{diag[\epsilon_1, \epsilon_2]; \epsilon_1\epsilon_2 = 1, |\epsilon_i| = 1, \epsilon_i \in C\}$,
$\mathbf{g}$ (resp. $\mathbf{g}_{(2)}$) : the Lie algebra of G (resp. $G_{(2)}$),
$\mathbf{h} = \mathbf{su}(2) \otimes I_2 + I_2 \otimes \mathbf{su}(2)$: the Lie algebra of H as a subspace of $\mathbf{g}$,
$\mathbf{t}$ (resp. $\mathbf{t}_{(2)}$) : the Lie algebra of T (resp. $T_{(2)}$),
$\mathbf{g}^c$ (resp. $\mathbf{t}^c$) : the complexification of $\mathbf{g}$ (resp. $\mathbf{t}$),
$diag[\epsilon_1, \epsilon_2, \cdots, \epsilon_n]$: a diagonal matrix
with diagonal elements $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$.

We give an $\mathrm{Ad}(G)$–invariant inner product $(\cdot, \cdot)$ on $\mathbf{g}$ by

$$(X, Y) = -B(X, Y) = -8\ Trace(XY), \quad (X, Y \in \mathbf{g}), \tag{2.1}$$

where B is the Killing form on $\mathbf{g}^c$. Let g be the G–invariant Riemannian metric on M induced from this inner product $(\cdot, \cdot)$. We denote by $e_j \in \sqrt{-1}\mathbf{t}^*$ (j=1,2,3,4), the Linear map

$$\sqrt{-1}\mathbf{t} \ni diag[x_1, x_2, x_3, x_4] \longrightarrow x_j \in C.$$

Put $\alpha_i = e_i - e_{i+1}$, $(i = 1, 2)$. We fix an lexicographic order $<$ on $\sqrt{-1}\mathbf{t}^*$ in such a way $e_1 > e_2 > e_3 > 0 > e_4$. The set $D(G)$ of all dominant G–integral forms is given by

$$D(G) = \{\lambda = \sum_{i=1}^{3} m_i e_i;\ m_1 \geq m_2 \geq m_3 \geq 0,\ \text{each } m_j \in Z\ \}.$$

On the other hand, the elements $H_{e_j} \in \sqrt{-1}\mathbf{t}$ such that $e_j(H) = B(H_{e_j}, H)$ for all $H \in \mathbf{t}^c$ are given as follows:

$$(2.2)\begin{cases} H_{e_1} = 1/32\ diag[3,-1,-1,-1],\ H_{e_2} = 1/32\ diag[-1,3,-1,-1], \\ H_{e_3} = 1/32\ diag[-1,-1,3,-1],\ H_{e_4} = 1/32\ diag[-1,-1,-1,3], \\ H_{\alpha_1} = 1/8\ diag[1,-1,0,0], \quad H_{\alpha_2} = 1/8\ diag[0,1,-1,0], \\ H_{\alpha_3} = 1/8\ diag[0,0,1,-1]. \end{cases}$$

Then the inner product $(\cdot,\cdot)$ induced on $\sqrt{-1}\mathbf{t}$ is given by

$$(2.3) \qquad (e_i, e_j) = (H_{e_i}, H_{e_j}) = \begin{cases} \frac{3}{32} & (i = j), \\ \frac{-1}{32} & (i \neq j), \end{cases}$$

where $i, j = 1,2,3,4$. The set P of all positive roots of $\mathbf{g}^c$ relative to $\mathbf{t}^c$ is

$$(2.4) \qquad P = \{e_i - e_j; 1 \leq i < j \leq 4\},$$

so we have

$$(2.5) \qquad \delta = 3e_1 + 2e_2 + e_3.$$

Therefore we have

$$(2.6) \qquad \begin{aligned} (\lambda + 2\delta, \lambda) &= (1/32)[(m_1 - m_2)^2 + (m_2 - m_3)^2 + (m_3 - m_1)^2 \\ &\quad + {m_1}^2 + {m_2}^2 + {m_3}^2 + 12m_1 + 4(m_2 - m_3)] \end{aligned}$$

for $\lambda = m_1e_1 + m_2e_2 + m_3e_3 \in D(G)$. Moreover, we have

$$(2.7) \qquad \begin{aligned} d(\lambda) &= \Pi_{1\leq i<j\leq 4} \frac{(e_i - e_j, \lambda + \delta)}{(e_i - e_j, \delta)} \\ &= (1/12)(m_1 + 3)(m_2 + 2)(m_3 + 1) \\ &\quad (m_1 - m_2 + 1)(m_2 - m_3 + 1)(m_1 - m_3 + 2) \end{aligned}$$

for $\lambda = m_1e_1 + m_2e_2 + m_3e_3 \in D(G)$. Here we have

Lemma 5. *Let* $\mathbf{m}$ *be the orthogonal complement of* $\mathbf{h}$ *in* $\mathbf{g}$ *with respect to the inner product* $(\cdot,\cdot)$. *Then* $\mathbf{m}$ *is given by*

$$(2.8) \quad \mathbf{m} = \{(A_{ij}) \in \mathbf{g};\ TraceA_{ij} = 0\ (i,j = 1,2),\ A_{11} + A_{22} = O_2\},$$

where O_2 *is the zero matrix of order* 2.

Proof. Since $\mathbf{h} = \{X \otimes I_2 + I_2 \otimes Y;\ X, Y \in \mathbf{g}_{(2)}\}$, $\mathbf{m}$ is perpendicular to $\mathbf{h}$. Moreover, $dim\mathbf{h} + dim\mathbf{m} = dim\mathbf{g}$. Hence, the proof of this Lemma is completed.

In the unitary irreducible representations of $G_{(2)}$, we use the same symbols as occured in the unitary irreducible representation of G. Let $V^{(2)}{}_l$ be a unitary irreducible representation space of $G_{(2)}$ with highest weight le_1, where $le_1 \in D(G_{(2)}) = \{me_1;\ m \geq 0, m \in Z\}$, [6, Th. 1, p. 46]. By the character formula of Weyl [11, pp. 332-333] for $\lambda = f_1e_1 + f_2e_2 + f_3e_3 \in D(G)$,

$$\chi_\lambda(h) = |\epsilon_i^{l_j}|/\xi(h) \tag{2.9}$$

for each $h = diag[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4] \in T$, where $|\epsilon_i^{l_j}|$ is the determinant of matrix of order 4 whose (i,j)–entries are $\epsilon_i^{l_j}$,

$$l_j = f_j + 4 - j \quad (j = 1, 2, 3), \text{ and } l_4 = 0, \tag{2.10}$$

and $\xi(h)$ is given as follows:

$$\xi(h) = \Pi_{1 \leq i < j \leq 4}(\epsilon_i - \epsilon_j). \tag{2.11}$$

Now let us consider the decomposition of V_λ, $(\lambda = \sum_{i=1}^4 f_ie_i \in D(G))$, into H–irreducible submodule as follows:

$$V_\lambda = \sum N(\lambda, l_1, l_2)\ V^{(2)}{}_{l_1} \otimes V^{(2)}{}_{l_2}, \tag{2.12}$$

where l_1, l_2 run over the set of all non-zero integers, $V^{(2)}{}_{l_1} \otimes V^{(2)}{}_{l_2}$ are irreducible representation spaces of $G_{(2)} \otimes G_{(2)}$, and $N(\lambda, l_1, l_2)$ is the multiplicity of $V^{(2)}{}_{l_1} \otimes V^{(2)}{}_{l_2}$ in V_λ.

We investigate $\lambda \in D(G)$ which belong to $D(G,H)$. $\lambda(\in D(G))$ belongs to $D(G,H)$ if and only if the unitary irreducible representation space V_λ of G contains $V^{(2)}{}_0 \otimes V^{(2)}{}_0$. We put

$$\begin{aligned} h = h_1 \otimes h_2 &= diag[x, x^{-1}] \otimes diag[y, y^{-1}] \\ &= diag[xy, xy^{-1}, x^{-1}y, x^{-1}y^{-1}] \\ &\in T_{(2)} \otimes T_{(2)} \subset T, \end{aligned}$$

then we have from (2,12)

$$\chi_\lambda(h) = \sum N(\lambda, l_1, l_2)\ \chi_{l_1}(h_1)\ \chi_{l_2}(h_2), \tag{2.13}$$

where χ_λ (resp. χ_{l_i}) is the character of the irreducible representation of G (resp. $G_{(2)}$) with the highest weight λ (resp. l_ie_1). Then we have

Lemma 6.

$$
\begin{aligned}
&(a) \quad V_{e_1} = {V^{(2)}}_1 \otimes {V^{(2)}}_1, \\
&(b) \quad V_{e_1+e_2} = {V^{(2)}}_2 \otimes V(2)_0 + {V^{(2)}}_0 \otimes {V^{(2)}}_2, \\
&(c) \quad V_{e_1+e_2+e_3} = {V^{(2)}}_1 \otimes {V^{(2)}}_1, \\
&(d) \quad V_{2e_1} = {V^{(2)}}_2 \otimes {V^{(2)}}_2 + {V^{(2)}}_0 \otimes {V^{(2)}}_0, \\
&(e) \quad V_{2e_1+e_2} = {V^{(2)}}_3 \otimes {V^{(2)}}_1 + {V^{(2)}}_1 \otimes {V^{(2)}}_3 + {V^{(2)}}_1 \otimes {V^{(2)}}_1, \\
&(f) \quad V_{2e_1+e_2+e_3} = {V^{(2)}}_2 \otimes {V^{(2)}}_2 + {V^{(2)}}_2 \otimes {V^{(2)}}_0 + {V^{(2)}}_0 \otimes {V^{(2)}}_2, \\
&(g) \quad V_{2e_1+2e_2+e_3} = {V^{(2)}}_3 \otimes {V^{(2)}}_1 + {V^{(2)}}_1 \otimes {V^{(2)}}_3 + {V^{(2)}}_1 \otimes {V^{(2)}}_1, \\
&(h) \quad V_{2e_1+2e_2+2e_3} = {V^{(2)}}_2 \otimes {V^{(2)}}_2 + {V^{(2)}}_0 \otimes {V^{(2)}}_0.
\end{aligned}
$$

Proof. Comparing with coefficients of both sides of (2.13) by using Weyl's character formular (2.9)-(2.11), we can obtain this Lemma.

Remark. Comparing with the dimensions of both sides in the decompositions in the above Lemma, we can check these decompositions.

Using (2.6), we get

Lemma 7.
(a) $(2\delta + 2e_1, 2e_1) = 4(\delta + e_1 + e_2 + e_3, e_1 + e_2 + e_3) = 9/8$,
(b) In case of $\lambda \in \{m_1e_1 + m_2e_2 + m_3e_3 \in D(G);\ m_1 \geq 3\}$,
$(2\delta + \lambda, \lambda) > (39/32)$.

Therefore, we find from Theorem 1, Lemma 6 and Lemma 7 that the least positive eigenvalue of the Laplace-Beltrami operator Δ_g of $(G/H, g)$ is 9/8.

We can easily obtain by a direct computation

$$
[\mathbf{m}, \mathbf{m}] \subset \mathbf{h}. \tag{2.14}
$$

Moreover, $SU(4)$ is simply connected. Hence, we get from Theorem 4 and the above facts

Lemma 8. *The normal homogeneous space $(G/H, g)$ is an irreducible symmetric Riemannian manifold.*

Thus, the proof of the main Theorem is completed.

From Theorem 2 and (2.1), we get

Lemma 9. *$(G/H, g)$ is an Einstein manifold with Einstein constant* 1/2.

Hence, we find from the main Theorem, Theorem 3 and Lemma 9 that the normal homogeneous space $(G/H, g)$ is stable.

Thus, the proof of Corollary is also completed.

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