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NOTE ON THE ANALYTIC CONTINUATION OF THE MULTIPLE HURWITZ ZETA FUNCTION $\zeta_n(s, a)$

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In this paper, we provide a new proof of the convergence and analytic continuation of the multiple Hurwitz zeta function $\zeta_n(s, a)$.

1. Introduction and Preliminaries

First we introduce the generalized zeta function (or the Hurwitz zeta function). The function $\zeta(s, a) = \sum_{n=0}^{\infty}(a+n)^{-s}$ is called the *generalized* (or *Hurwitz*) zeta function, where s is a complex number and $a > 0$. It can be shown that $\zeta(s, a)$ is an analytic function for Re$s > 1$. In particular, when $a = 1$, $\zeta(s, 1) = \sum_{n=1}^{\infty} n^{-s} = \zeta(s)$ is the well-known *Riemann* zeta function. Moreover by the contour integral representation $\zeta(s, a)$ can be continued to a meromorphic function having only simple pole at s=1 with its residue 1 [5]. In [1], E.W.Barnes introduces the mutiple Hurwitz ζ-function, for Re$s > r$,

$$\zeta_r(s, a \mid w_1, w_2, \ldots, w_r) = \sum_{m_1, m_2, \ldots, m_r = 0}^{\infty} \frac{1}{(a+\Omega)^s},$$

where $\Omega = m_1 w_1 + m_2 w_2 + \ldots + m_r w_r$. He also represents the r-ple Hurwitz ζ-function by the contour integral

$$\zeta_r(s, a \mid w_1, w_2, \ldots, w_r) = \frac{i\Gamma(1-s)}{2\pi} \int_L \frac{e^{-az}(-z)^{s-1}}{\prod_{k=1}^{r}(1-e^{-w_k z})} dz$$

where the conditions for a and $w_1, \ldots, w_r$, the possible contour L are given in [1], Γ is the well-known gamma function which has simple poles at $z = 0, -1, -2, \ldots$; and $\text{Res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}$, $n = 0, 1, 2, \ldots$ [5].

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Here we restrict these when $w_k = 1, k = 1, 2, \ldots, n$, $a > 0$ and C the contour given as in [5], p. 245.

That is to say, we get

$$\zeta_n(s,a) = \sum_{k_1,k_2,\ldots,k_n=0}^{\infty} (a + k_1 + k_2 + \ldots + k_n)^{-s},$$

which is an analytic function for Re$s > n$ by the Eisenstein's Theorem [2], [3]. Furthermore $\zeta_n(s,a)$ can be continued to a meromorphic function with only poles at $s = 1, 2, \ldots, n$. For by the contour integral representation

$$\zeta_n(s,a) = \frac{i\Gamma(1-s)}{2\pi} \int_C \frac{e^{-az}(-z)^{s-1}}{(1-e^{-z})^n} dz$$

the integral is valid for $a > 0$ and all s so $\zeta_n(s,a)$ has possible poles only at the poles of $\Gamma(1-s)$ i.e., $s = 1, 2, 3, \ldots$. But by the series definition $\zeta_n(s,a)$ is holomorphic for Re$s > n$. In particular when $n = 1$, $\zeta_1(s,a) = \sum_{k=0}^{\infty}(a+k)^{-s} = \zeta(s,a)$ is the well known Hurwitz ζ-function.

2. The analytic continuation of $\zeta_n(s,a)$

In fact to show the analytic continuation of $\zeta_n(s,a)$ through the contour integral representation, the complicated theory and computation are required [1]. In this section we reduce the multiple Hurwitz zeta function to the generalized zeta function and explain the analytic continuation of $\zeta_n(s,a)$ by using that of the generlized zeta function $\zeta(s,a)$.

Lemma 2.1. *The multiple Hurwitz zeta function*

$$\zeta_n(s,x) = \sum_{k=0}^{\infty} \frac{\binom{k+n-1}{n-1}}{(x+k)^s}$$

where $\binom{k+n-1}{n-1}$ *is the combination notation.*

Proof. Observe the series definition of

$$\zeta_n(s,x) = \sum_{k_1,\ldots,k_n=0}^{\infty} (x + k_1 + k_2 + \ldots + k_n)^{-s}.$$

Then the number of solutions of $k_1 + k_2 + \ldots + k_n = k$, $k = 0, 1, 2, \ldots$ $(k_1, k_2, \ldots, k_n) \in \mathrm{N}^n$ is equal to the coefficient of x^k in the expansion of the

Maclaurin series of $(1-x)^{-n}$ i.e., $\binom{-n}{k}$. By the definition of the generalized combination notation,

$$\begin{aligned}\binom{-n}{k} &= \frac{-n(-n-1)\dots(-n-k+1)}{k!}\\ &= (-1)^k\frac{(n+k-1)\dots(n+1)n}{k!}\\ &= (-1)^k\frac{(n+k-1)!}{(n-1)!k!}\\ &= (-1)^k\binom{k+n-1}{n-1}.\end{aligned}$$

On the other hand the Maclaurn series expansion of $(1-x)^{-n}$ is

$$\sum_{k=0}^{\infty}\binom{-n}{k}(-x)^k = \sum_{k=0}^{\infty}\binom{k+n-1}{n-1}x^k.$$

Theorem 2.2. *The multiple Hurwitz zeta function $\zeta_n(s,a)$ can be continued to a meromorphic function with simple poles only at $s = 1, 2, \dots, n$.*

Proof. By Lemma 2.1,

$$\zeta_n(s,x) = \sum_{k=0}^{\infty}\binom{k+n-1}{n-1}\Big/(x+k)^s.$$

Consider

$$\begin{aligned}\binom{k+n-1}{n-1} &= \frac{(k+n-1)(k+n-2)\dots(k+2)(k+1)}{(n-1)!}\\ &= \sum_{i=0}^{n-1}Q_{n,i}k^i,\end{aligned}$$

where $Q_{n,i}$ are rational numbers, $i = 0, 1, 2, \dots, n$,

$$Q_{n,n-1} = \frac{1}{(n-1)!} \quad\text{and}\quad Q_{n,0} = 1.$$

Then we have

$$\zeta_n(s,x) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{n-1}Q_{n,i}k^i\right)\Big/(x+k)^s.$$

Consider, by the binomial theorem,

$$k^i = \{(-x) + (x+k)\}^i = \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} (x+k)^j.$$

Therefore we have

$$\zeta_n(s,x) = \sum_{k=0}^{\infty} \left[\sum_{i=0}^{n-1} Q_{n,i} \left(\sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} (x+k)^j \right) \right] \Big/ (x+k)^s. \qquad (1)$$

Consider

$$\sum_{i=0}^{n-1} Q_{n,i} \left(\sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} (x+k)^j \right) = \sum_{l=0}^{n-1} P_l(x)(x+k)^l,$$

where $P_l(x)$ is a polynomial in x with rational coefficients. Therefore we have

$$\begin{aligned} \zeta_n(s,x) &= \sum_{l=0}^{n-1} P_l(x) \sum_{k=0}^{\infty} \frac{1}{(x+k)^{s-l}} \qquad (2) \\ &= \sum_{l=0}^{n-1} P_l(x) \zeta(s-l,x). \end{aligned}$$

We know that $\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^s}$ converges absolutely for $\mathrm{Re}s > 1$. So we see that $\zeta_n(s,x)$ converges absolutely for $\mathrm{Re}(s-l) > 1, \quad l = 0,1,2,\ldots,n-1$. Thus we conclude that $\zeta(s,x)$ converges absolutely for $\mathrm{Re}s > n$.

Furthermore in the first section we know that $\zeta(s,a)$ can be continued to a meromorphic function with only simple pole at $s=1$. So we deduce that $\zeta_n(s,x)$ can be continued to a meromorphic function with only simple poles at $s-l=1, \quad l=0,1,2,\ldots,n-1$, i.e., $s=1,2,\ldots,n$.

In particular, when $n = 2, 3$, we have

Corollary 2.3. *We have*

$$\begin{aligned} \zeta_2(s,x) &= \zeta(s-1,x) + (1-x)\zeta(s,x), \\ \zeta_3(s,x) &= \frac{1}{2}(\zeta(s-2,x) + (3-2x)\zeta(s-1,x) + (x^2-3x+2)\zeta(s,x)). \end{aligned}$$

Proof. Replacing n by 2 and 3 in (2.1), we can have these.

Corollary 2.4. *For* $k = 1, 2, \ldots, n$,

$$\operatorname{Res}_{s=k}\zeta_n(s,x) = p_{k-1}(x),$$

where $p_l(x)$ *is the polynomial in* x *appearing in* (2.2).

Proof. By Theorem 2.2, $\zeta_n(s,x)$ has only simple poles at $s = 1, 2, \ldots, n$. From (2.2) we have

$$\zeta_n(s,x) = \sum_{l=0}^{n-1} p_l(x)\zeta(s-l,x).$$

Since $\zeta(s,x)$ has only simple pole at $s = 1$ with residue 1,

$$\begin{aligned}
\operatorname{Res}_{s=k}\zeta_n(s,x) &= \lim_{s\to k}(s-k)\zeta_n(s,x) \\
&= p_{k-1}(x)\lim_{x\to k}(s-k)\zeta(s-k+1,x) \\
&= p_{k-1}(x)\operatorname{Res}_{s=k}\zeta(s-k+1,x) \\
&= p_{k-1}(x),
\end{aligned}$$

where $k = 1, 2, \ldots, n$.

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