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SUPERCOMPACTNESS, PRODUCTS AND THE AXIOM OF CHOICE

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It is shown that the Axiom of Choice holds in **ZF** iff the product of any supercompact spaces is supercompact, while the same result for T_0-spaces and its analogue for frames can be proved without Choice.

Recall that a topological space X is called *supercompact* if it has only trivial covers, that is, every open cover of X actually contains X. Obviously, this means that X has a largest proper open subset.

We note that such spaces do in fact occur widely. Thus for any partially ordered set P with zero(=smallest element), the collection $\mathcal{U}P$ of all *upsets* $W \subseteq P(x \geq y \in W$ implies $x \in W)$ is a supercompact topology on P, and hence any subtopology of $\mathcal{U}P$ will be supercompact. This covers, for instance, all injective T_0-spaces, that is, continuous lattices equipped with the topology of Scott open set [2]. These, in turn, include all Sierpinski cubes, and consequently every T_0-space is a subspace of a supercompact T_0-space. Alternatively, any space X determines a space $\hat{X}$ obtained by adding a new point to X such that X is an open subspace of $\hat{X}$ and the only neighbourhood of the new point is the total space; evidently, $\hat{X}$ is supercompact.

At the same time, supercompactness is rather an extreme form of compactness, especially since a supercompact space must be fairly unseparated: such a space may be T_0 but if it is T_1 it is a singleton or empty. In view of this, it is noteworthy that, in Zermelo-Fraenkel set theory, the foundational position of supercompactness is the same as that established by Kelley [4] for compactness.

The Axiom of Choice holds iff the product of any supercompact spaces is supercompact.

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Given supercompact spaces $X_i (i \in I)$ with largest proper open subsets $S_i \subseteq X_i$, and $X = \Pi X_i$ with the projections $p_i : X \to X_i$, put $T = \cup\{p_i^{-1}[S_i] | i \in I\}$. We claim this is the largest proper open subset of X.

Let, then, W be any proper open subset of X and consider any $U \subseteq W$ such that

$$U = p_{i_1}^{-1}[U_1] \cap \cdots \cap p_{i_n}^{-1}[U_n]$$

for open $U_k \subseteq X_{i_k}$; Since U is also proper, $U_k \neq X_{i_k}$ for some k, hence $U_k \subseteq S_{i_k}$, and thus $U \subseteq p_{i_k}^{-1}[S_{i_k}] \subseteq T$. This shows that $W \subseteq T$ since W is the union of such U. To see that $T \neq X$, choose $c_i \notin S_i$ for each $i \in I$: then $c = (c_i)_{i \in I} \notin T$.

Conversely, for any family $(E_i)_{i \in I}$ of non-void sets, take any element $* \notin \cup E_i$ and define spaces X_i with underlying sets $E_i \cup \{*\}$ and open sets $\emptyset$, $\{*\}$, and $E_i \cup \{*\}$. Obviously these X_i are supercompact, and hence $X = \Pi X_i$ is supercompact by hypothesis. Now, put $W_i = p_i^{-1}\{*\}$, and for $x \in E_i$ let $\tilde{x} \in X$ be the point such that $\tilde{x}_i = x$ and $\tilde{x}_j = *$ for $j \neq i$. Then $\tilde{x} \notin W_i$ and therefore $\cup W_i \neq X$. It follows that, $\mathbf{C}$ meaning complement,

$$\begin{aligned} \emptyset &\neq \mathbf{C}(\cup W_i) = \cap \mathbf{C} W_i = \cap \mathbf{C} p_i^{-1}(*) \\ &= \cap p_i^{-1}[E_i] = \Pi E_i, \end{aligned}$$

proving the Axiom of Choice.

We note that the spaces involved in the second part of this proof will, in general, be badly non-T_0. There is a good reason for this, namely:

Any product of supercompact T_0-spaces is supercompact.

Going back to the place in the first part of the above proof where the Axiom of Choice is invoked by picking $c_i \notin S_i$ in X_i for each $i \in I$, consider any $x \notin S_i$ in X_i. Then $S_i \subseteq \mathbf{C}\overline{\{x\}}$ and thus $S_i = \mathbf{C}\overline{\{x\}}$, so that, for any $x, y \notin S_i$ in X_i, $\overline{\{x\}} = \overline{\{y\}}$ and therefore $x = y$ for T_0-spaces. Thus, there is only one point in X_i outside S_i, for each $i \in I$ in the present situation, and this eliminates the need for the Axiom of Choice at this stage.

The last result leads to a further condition equivalent to the Axiom of Choice. For any space X, let X^0 be the T_0-reflection of X, that is, the quotient space of X obtained by identifying points with equal neighbourhood filters. Obviously X and X^0 have isomorphic lattices of open sets so that X^0 is supercompact iff X is. Now we have, in Zermelo-Fraenkel set theory:

The Axiom of Choice holds iff any product space ΠX_i is supercompact whenever ΠX_i^0 is supercompact.

Given the Axiom of Choice and $(X_i)_{i \in I}$ such that $Y = \Pi X_i^0$ is supercompact, it follows that each X_i^0 is supercompact since the projections $p_i : Y \to X_i^0$ induce frame embeddings (=preserving $\bigcap$ and $\bigcup$) $U \rightsquigarrow p_i^{-1}[U]$ for the open sets U of X_i^0. Hence each X_i, and therefore ΠX_i, is supercompact, the latter by our first result.

For the converse, it suffices to deduce that ΠX_i is supercompact for any family of supercompact spaces, but this follows immediately from the fact that ΠX_i^0 is supercompact, by our second result.

An alternative to the first part of this proof would be to use the result of Banaschewski [1] that the Axiom of Choice is equivalent to the condition that $(\Pi X_i)^0 = \Pi X_i^0$ for any family of spaces.

Finally we note that, again in analogy with compactness but considerably easier to prove, we have:

Any coproduct of supercompact frames is supercompact.

Recall, first, that a *frame* is a complete lattice L in which $x \wedge \bigvee S = \bigvee \{x \wedge t | t \in S\}$ for all $x \in L$ and $S \subseteq L$, and a *frame homomorphism* is a map $h : L \to M$ between frames which preserves all finitary meets, including the unit e, and arbitrary joins, including the zero 0. Any family $(L_i)_{i \in I}$ of frames has a *coproduct* $\oplus L_i$, and *supercompactness*, naturally, means there exists a largest element strictly smaller than the unit. For general facts about frames see Johnstone [3].

Let, then, $L = \oplus L_i$ with coproduct maps $k_i : L_i \to L$ and assume each L_i is supercompact, with $s_i \in L_i$ the largest element smaller than the unit. Note that, as a consequence of this, each $k_i : L_i \to L$ has a left inverse $\ell_i : L \to L_i$, defined such that

$$\ell_i k_i = id_{L_i}, \ell_i k_j(x) = \begin{cases} e & (x \not\leq s_j) \\ 0 & (x \leq s_j) \end{cases} \quad \text{for } j \neq i.$$

Now let $s = \bigvee \{k_i(s_i) | i \in I\}$. We claim this is the largest element of L smaller than the unit.

To begin with, indeed $s < e$ since $s = e$ implies, for any $j \in I$,

$$e = \ell_j(e) = \bigvee \{\ell_j k_i(s_i) | i \in I\} = s_j,$$

a contradiction. Further, for any $a < e$, if

$$x = k_{i_1}(x_1) \wedge \cdots \wedge k_{i_n}(x_n) \leq a$$

for some $x_k \in L_{i_k}$ then $x \leq s$ by the same argument as in our first proof, and since a is the join of these x we have $a \leq s$, as desired.

In conclusion, we remark that supercompact frames are as ubiquitous for frames as supercompact spaces are for spaces: For any frame L, "adding a new top" produces a supercompact frame $\hat{L}$ such that $L \cong \{x \in \hat{L} | x \leq s\}$ for the largest $s < e$ of $\hat{L}$. This makes every frame an "open" quotient of a supercompact frame. In particular, it follows that there is a large supply of supercompact frames that are not isomorphic to a topology.

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