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# BIACCESSIBLE POINTS

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A point is *biaccessible* if it can be approached by two essentially different sequences. A topological space is *biaccessible* if it has a dense subset of biaccessible points. The elementary properties of such points and spaces are established while some of their quirks are revealed in examples. We characterize Kunen's *sequentially small* spaces as those which do not contain an embedded copy of $\beta\mathbf{N}$, and we relate the above to *sequentially pinched* spaces, those Hausdorff spaces having no closed subspace which is homeomorphic to an infinite subspace of $\beta\mathbf{N}$. We conclude with two questions.

## 1. Introduction

$T$ denotes an arbitrary (no built-in separation axioms) topological space throughout.

By a *discrete sequence* we mean a sequence $(t_n)$ of distinct points of $T$ whose subspace topology is discrete.

Two extremes of "accessibility" of a point $t \in T$ are:

(I) A discrete sequence converges to $t$.

(II) No discrete sequence has $t$ as a cluster point.

Isolated points are certainy of type II: so are *weak P-points*-points which are not accumulation points of any countable subset ([9], [12]). As weak $P$-points of $\mathbf{N}^* = \beta\mathbf{N}\backslash\mathbf{N}$ exist ([12]), they constitute a dense subset of nonisolated points of $\mathbf{N}^*$ ([6, 6S.6]) of type II. Nonisolated points of first countable Hausdorff spaces are of type I since they are limits of discrete sequences.

In this paper we study an accessiblility property (Def. 1.1) which lies between I and II. Interest in biaccessiblity arose from a condition

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concerning an automatic continuity result in analysis ([1. p.267, condition (*)]).

The first infinite and first uncountable ordinals are denoted by $\omega$ and $\omega_1$, respectively. If every neighborhood of a point $t$ contains infinitely many points of a set $E$ then $t$ is called an *$\omega$-accumulation point* of $E$. (In $T_1$-spaces the notions of accumulation point and $\omega$-accumulation point coincide.) A point $t$ is a cluster point of a sequence $(t_n)$ of distinct points iff $t$ is an $\omega$-accumulation point of the set $\{t_n\}$ ([8]).

If a subset $E$ of $T$ is closed whenever $E$ contains each limit of every convergent sequence of points of $E$ then $T$ is called a *sequential space* ([4, p.78]). For terms not defined here, see [4], [6] or [8].

**Definition 1.1.** (a) We say that $T$ is *biaccessible at $t$* or that $t$ is a *biaccessible point of $T$* if $t$ is a common accumulation point of two disjoint countably infinite subsets $D$ and $E$ whose union $D \cup E$ is discrete in its subspace topology. The set of biaccessible points of $T$ is denoted $bi\ T$.

(b) We say that $T$ is a *biaccessible space* if it is biaccessible at each point of a dense subset.

Limits of discrete sequences are biaccessible: if $t_n \to t$ then every subsequence of $t_n$ has $t$ as a cluster point. A nonisolated point $t$ of a sequential space $T$ is a limit of a sequence of points distinct from $t$: if $T$ is Hausdorff, then $t$ is a biaccessible point of $T$. A Hausdorff space without isolated points (a) which is sequential or (b) which has a dense subset of first countable points is a biaccessible space. On the othere hand, no point $t$ of $\beta N$ is biaccessible - If $t$ is a cluster point of a discrete sequence $(t_n)$ in $\beta N$ then at most one of *any two* disjoint subsequences of $(t_n)$ has $t$ as a cluster point (2.5)- so $\beta N$ is not a biaccessible space (cf. 1.2b).

**Examples 1.2.** (a) For $t \in \beta\mathbf{N}\backslash\mathbf{N}$, equip $\mathbf{N} \cup \{t\}$ with its subspace topology; $t$ is the unique accumulation point of the discrete sequence $(n)$ and is not biaccessible.

(b) There exist points $p \in \beta\mathbf{N}\backslash\mathbf{N}$ which are accumulation points of discrete sequences $(p_n)$ from $\beta\mathbf{N}\backslash\mathbf{N}$; any such $p$ is also an accumulation point of the discrete sequence $(n)$. However, $\{p_n\} \cup \{n\}$ is clearly not discrete.

(c) The point $\omega$ is the only biaccessible point in the ordinal space $\omega+1$.

(d) No point of a discrete or an indiscrete space is biaccessible.

(e) Let $\mathbf{N}$ be endowed with the $T_0$-topology generated by the lower segments $\{k : k < n\}$, $n \in \mathbf{N}$. It is first countable and neither discrete nor indiscrete. As it has no infinite discrete subsets (the only discrete subset is $\{1\}$), it is devoid of biaccessible points.

(f) $\mathbf{N}$ with the cofinite topology is a sequential $T_1$-space. It has no infinite discrete subsets, hence no biaccessible points.

(g) Let $\mathbf{R}^+$ be the nonnegative reals. The *long line* ([6, 16H], [4,3.12.18]) $T = \omega_1 \times \mathbf{R}^+$ in its lexicographic order is a biaccessible space. $T$ with $\omega_1$ adjoined as the greatest element is the *long segment* and $w_1$ is the only non-biaccessible point in this compact Hausdorff space; it is first countable at all other points.

Dow and Vaughan have investigated biaccessible points in contrasequential spaces in [3].

## 2. A sequential characterization; stability properties

The following characteristics of biaccessibility and biaccessible spaces follow readily.

**2.1** (a) $t \in bi\,T$ iff there exists a discrete sequence $(t_n)$ such that $t$ is a cluster point of $(t_{2n})$ and $(t_{2n-1})$.

(b) If $t \in bi\,T$ then, for any neighborhood $U$ of $t$, $t \in bi\,U$ (in $U$'s subspace topology). Therefore any open subspace of a biaccessible space is a biaccessible space.

(c) If $S$ is a subspace of $T$ then $bi\,S \subset bi\,T$.

(d) $t \in bi\,T$ iff there is some countable subspace $S$ of $T$ such that $t \in bi\,S$.

(e) $T$ is a biaccessible space iff, for each nonempty open subspace $U$, $bi\,U$ is nonempty; indeed, $T$ is biaccessible iff each nonempty open subspace is biaccessible.

If each point of $T$ has a neighborhood that is a biaccessible space we might say that $T$ is "locally biaccessible"; 2.1e, however, shows that a space is locally biaccessible iff it is biaccessible.

The following theorem shows that every $T$ decomposes uniquely into disjoint subsets $F$, $U$ and $A$ where $F$ is the largest closed biaccessible subspace of $T$, $U$ is the largest open subset consisting solely of non-biaccessible points and the points of the remainder $A$ lie in the closure of $\{t \in bi\,T\colon t$ does not belong to any biaccessible subspace of $T\}$.

**2.2** (a) *If a subspace $S$ of $T$ is a biaccessible space then so is its closure* *cl* $S$.

(b) *The union* $F$ *of all biaccessible subspaces of* $T$ *is closed and is the largest biaccessible subspace of* $T$.

(c) *The largest open set* $U$ *disjoint from* $bi\,T$ *is a dense subset of* $T\backslash F$.

*Proof.* (a) Use 2.1c.

(b) If $D_i$ is dense in $T_i$ then $\cup_i D_i$ is dense in $\cup_i T_i$ so (b) follows from (a) and 2.1c.

(c) Let $\mathcal{V} = \{V \subset T : V \text{ is open and } V \cap bi\,T = \emptyset\}$. The open set $U = \cup\mathcal{V}$ is the largest element of $\mathcal{V}$. Let $W$ be a nonempty open subset of $T\backslash F$. $W$ is not a biaccessible subspace so $W$ contains a nonempty open subset $V$ containing no biaccessible point of $W$, hence none of $T$ either by 2.1b. Thus, $V \in \mathcal{V}$; therefore, $T\backslash F \subset cl\,U$. If $F$ meets $U$ then some biaccessible subspace $S$ of $T$ meets $U$. Therefore $U$ contains a biaccessible point of $S$ which, by 2.1c, must be a biaccessible point of $T$ as well. As this contradicts the way $\mathcal{V}$ is defined, it follows that $F$ does not meet $U$, i.e., that $U \subset T\backslash F$.

**Products-Examples 2.3.** (a) The discrete doubleton space 2 has no biaccessible points, yet each point of the Cantor discontinuum $H$ is biaccessible and $H$ is homeomorphic to the product $2^\omega$.

(b) No point of $\mathbf{N}^* = \beta\mathbf{N}\backslash\mathbf{N}$ is biaccessible (2.5), yet $N^* \times N^*$ is a biaccessible space (4.1).

The previous two observations suggest that even though there may be a paucity of biaccessible points in component spaces, this poverty need not carry over to their product. The following theorem confirms this suspicion.

**Product theorem 2.4.** (a) *The product of a biacessible space* $T$ *with any topological space* $S$ *is a biaccessible space.*

(b) *Every point of the product* $T$ *of an infinite family* $\{T_b : b \in B\}$ *of nonsingleton* $T_1$*-spaces is biaccessible; therefore the product* $T$ *is a biaccessible space.*

(c) *If the* $T_1$*-spaces* $S$ *and* $T$ *have dense subsets consisting of accumulation points of discrete sequences then* $T \times S$ *is biaccessible.*

*Proof.* (a) Any basic open subset $U \times V$ of $T \times S$ contains a point $(t, s)$ with $t \in bi\,T$. Let $(t_n)$ be a discrete sequence from $T$ such that $t$ is a cluster point of the alternate sequences $(t_{2n})$ and $(t_{2n-1})$; $(t_n, s)$ is a discrete sequence in $T \times S$ and $(t, s)$ is a cluster point of $(t_{2n}, s)$ and

$(t_{2n-1}, s)$.

(b) By (a), it suffices to consider $B = \mathbf{N}$. Let $t = (t_n) \in T$. For each $n$, choose $s_n \in T_n$ different from $t_n$. For each $k \in \mathbf{N}$, let $r_k \in T$ have $s_k$ as its $k$-th coordinate and be $t_n$ at all the rest. Obviously $r_n \to t$. To see that $(r_n)$ is a discrete sequence, we argue as to why $r_1$ is isolated. Choose a neighborhood $U_1$ of $s_1$ in $T_1$ that does not contain $t_1$. $(U_1 \times \prod_{n \geq 2} T_n) \cap \{r_n\} = \{r_1\}$. As each point of $T$ is biaccessible, $T$ is a biaccessible space.

(c) We show that each basic open subset $U \times V$ of $T \times S$ contains a biaccessible point. Each such neighborhood contains a point $(t, s)$ where $t$ and $s$ are accumulation points of discrete sequences $(t_n)$ and $(s_n)$, respectively, in $T$ and $S$. For each $n$, let $r_{2n-1} = (t_n, s)$ and $r_{2n} = (s, t_n)$; $(t, s)$ is clearly a cluster point of $(r_{2n-1})$ and $(r_{2n})$. The argument that $(r_n)$ is a discrete sequence is based on arguments such as the following: There is a neighborhood $U$ in $T$ that contains $t_1$ and no other $t_n$ or $t$ since $T$ is $T_1$. $(U \times S) \cap \{r_n\} = \{(t_1, s)\} = \{r_1\}$.

Note that any countably compact $T_3$-space without isolated points fulfills the requirements of 2.4c.

**2.5**. *Let $E$ be a countable discrete subset of the Tychonov space $T$ and consider the following three statements:*

(a) *Disjoint subsets of $E$ do not share accmulation points.*

(b) *$cl_T E$ is homeomorphic to a subspace of $\beta\mathbf{N}$.*

(c) *$cl_T E$ is homeomorphic to $\beta\mathbf{N}$.*

Statements (a) and (b) are equivalent while each is implied by (c); if $T$ is compact and $E$ is infinite then all three are equivalent.

*Proof.* (a) $\Rightarrow$ (b). Every subset of discrete space $E$ is a zeroset of $E$. As disjoint subsets of $E$ have disjoint closure in $cl_T E$, then $cl_T E \subset \beta E \subset \beta\mathbf{N}$ ([6, 6.7(6)]).

(b) $\Rightarrow$ (a) Suppose $i : cl_T E \to \beta\mathbf{N}$ is a homeomorphism (into). The countable set $i(E)$ is $C^*$-embedded in $\beta\mathbf{N}$ (6, 6O(6)]). Therefore disjoint subsets of the discrete subset $i(E)$ cannot have common accumulation points in $\beta N$, hence none in $i(cl_T E)$ either. This reflects back into the desired statements about $E$ and $cl_T E$ in $T$.

Finally, (c) $\Rightarrow$ (b) is trivial and, for $E$ infinite and $T$ compact, (a) implies that $E \subset cl_T E \subset \beta E \subset \beta N$ so $cl_T E$ is a copy of $\beta\mathbf{N}$.

## 3. Sequentially Small Spaces

Following Kunen [9], we call a topological space $T$ *sequentially small* if each infinite subset $A$ of $T$ contains an infinite subset $C$ whose closure does not contain a copy of $\beta\mathbf{N}$. Sequentially compact spaces are sequentially small and so are metric spaces. The following theorem provides a succinct characterization of sequentially small spaces.

**3.1**. *$T$ is sequentially small iff $T$ does not contain a copy of $\beta\mathbf{N}$.*

*Proof.* If $T$ contains no copies of $\beta\mathbf{N}$ it is obviously sequentially small. Conversely, suppose that $\beta\mathbf{N}$ is a subspace of $T$. Since the closure of any infinite subspace of $\beta\mathbf{N}$ contains a copy of $\beta\mathbf{N}$ [6, 6O(6)], $T$ is not sequentially small.

The following notion is generally stronger than sequentially small: the things coincide in compact Hausdorff spaces (3.3a, 3.5).

**Definition 3.2**. A Hausdorff space $T$ is *sequentially pinched* if no closed subspace of $T$ is homeomorphic to an infinite subspace of $\beta\mathbf{N}$.

**Examples 3.3**. (a) As a sequentially pinched space does not contain a closed copy of $\mathbf{N}$, it is countably compact. As every topological space of cardinality $< 2^{\mathfrak{c}}$ is sequentially small, $\mathbf{N}$ is sequentially small but not sequentially pinched.

(b) A sequentially compact Hausdorff space $T$ is sequentially pinched: An infinite closed subspace $F$ of such a space must contain a discrete sequence and its limit; by 2.5, no such space can be homeomorphic to a subspace of $\beta\mathbf{N}$.

(c) The Tychonoff plank $T$ ([6, pp. 123-125]) contains a closed copy of $\mathbf{N}$ so it is not sequentially pinched. However, $\beta T = (\omega_1 + 1) \times (\omega + 1)$ is sequentially pinched since it is sequentially compact.

**3.4**. *A sequentially pinched Tychonov space $T$ without isolated points is a biaccessible space.*

*Proof.* Let $U$ be a nonempty open set and let $V$ be a nonempty open subset of $U$ whose closure is contained in $U$. Since $V$ is an infinite subset of a Hausdorff space, it has an infinite discrete subset $E$. Since $T$ is sequentially pinched, $cl\, E$ is not homeomorphic to any subspace of $\beta\mathbf{N}$. It follows from 2.5 that $E$ has disjoint subsets with a common accumulation point $t$. Now $t$ is a biaccessible point of $T$ and is in $cl\, E \subset cl\, V \subset U$.

**3.5**. *If a Hausdorff space $T$ is sequentially pinched then it is sequentially* *small. The converse holds if $T$ is a compact Hausdorff space.*

*Proof.* As a copy of $\beta\mathbf{N}$ in a Hausdorff space is closed, a sequentially pinched space cannot contain $\beta\mathbf{N}$. By 3.1, $T$ is sequentially small.

Conversely, suppose that $S = cl_T S$ is homeomorphic to an infinite subspace of $\beta\mathbf{N}$ in the compact Hausdorff space $T$. Since $S$ is compact, it is a copy of an infinite closed subspace of $\beta\mathbf{N}$. As such, it must contain a copy of $\beta\mathbf{N}$ by [6, 6O(6)]. By 3.1, $T$ is not sequentially small.

The product of any nonempty biaccessible space, $[0,1]$ say, with $\beta\mathbf{N}$ is a biaccessible space by 2.4a. It is clearly neither sequentially pinched nor sequentially small.

## 4. Further examples and a question

Example 4.1 was suggested by Dow [2] and led to 2.4c. It is a biaccessible compact Hausdorff space with a dense set of non-biaccessible points.

**Example 4.1.** By [4, 3.6.15], no nontrivial sequence in $\mathbf{N}^* = \beta\mathbf{N}\setminus\mathbf{N}$ converges; hence, the same is true for the compact Hausdorff space $T = \mathbf{N}^* \times \mathbf{N}^*$. The weak $P$-points of $N^*$ are dense and so $\{(p,q) : p$ and $q$ are weak $P$-points $\}$ is a dense set of non-biaccessible points of $T$. $T$ is a biaccessible space by 2.4c and the remark after the theorem.

Gillman and Jerison ([6, 9.15]) construct a countably compact subspace $G$ of $\beta N$ whose square $G \times G$ is not pseudocompact. Their construction is modified in 4.2 to produce a sequentially small subspace $G$ of $\beta\mathbf{N}$ which is a pseudocompact and not countably compact space but whose square is sequentially small and not pseudocompact. Sequentially pinched spaces, by contrast, are countably compact by 3.3a.

**Example 4.2.** In [6, 9.15] use $\sum = \{S \subset \mathbf{N} : S$ is infinite $\}$. For any $S \in \sum$, the clopen set $cl_{\beta\mathbf{N}}S$ is homeomorphic to $\beta\mathbf{N}$ and therefore has cardinality greater than $c$. As the weak $P$-points of $\mathbf{N}^*$ are dense, the set of those contained in $cl_{\beta\mathbf{N}}S\setminus S$ has cardinality at least $c$. Thus, the points $p_S$ chosen in [6, 9.15] may be assumed to be weak $P$-points, so as to produce a set of $c$ weak $P$-points $P = \{p_S : S \in \sum\}$ of $\mathbf{N}^*$. As noted in [6], $G = N \cup P$ has a square $G \times G$ which is not pseudocompact. As card $G <$ card $\beta\mathbf{N}$, $G$ and $G \times G$ are both sequentially small. Clearly neither is sequentially pinched.

$G$ is pseudocompact ([7]). If $C$ is a denumerable subset of $P$ then the closure of $C$ in $G$ is contained in $P$. Consequently any potential accumulation point of $C$ in $G$ is a weak $P$-point of $N^*$. $C$ is therefore closed in $G$ and $G$ is not countably compact.

Kunen [9] notes a lemma due to Malykhin [10]: If $\beta\mathbf{N}$ is embeddable in a product $\prod_{n\in\mathbf{N}} T_n$ of compact Hausdorff spaces $T_n$ then $\beta\mathbf{N}$ is embeddable in at least one $T_n$. Thus, a countable product of sequentially small compact Hausdorff spaces is sequentially small.

**Question 4.3.** What products of sequentially pinched spaces are sequentially pinched?

In 4.3 the number of factors needs to be limited. Let $K$ be the collection of characteristic functions on $\mathbf{N}$. Since $\beta\mathbf{N}$ is embedded in $2^K$, $\beta N$ is embedded in any product $\prod_{s<t} X_s$ where $t \geq c = |K|$ and each $X_s$ contains at least two points. Moreover, Shapirovskii ([11, Cor.3]) answers 4.3 for $t$-many compact Hausdorff factor spaces where $t < cf(c)$, the cofinality of $c$.

Fedorcuk ([5]) shows that the existence of a compact Hausdorff space of the cardinality of the continuum with no nontrivial convergent sequences is consistent with $ZFC$. Such a space is sequentially pinched since it cannot contain $\beta\mathbf{N}$.

**Question 4.4.** Without additional axioms, is there an infinite regular sequentially pinched space with no nontrivial convergent sequences?

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## References

[1] E. Beckenstein, L. Narici and A.R. Todd, *Automatic continuity of linear maps in spaces of continuous functions*, Manuscripta Math. 62(1988), 257-275.

[2] A. Dow, personal communication.

[3] A. Dow and J. Vaughan, *Accessible and biaccessible points in contrasequential spaces*, Bull. N.Y. Acad. Sci., to appear.

[4] R. Engelking, General Topology, Polish Sci. Publishers, Warszawa 1977.

[5] V.V. Fedorcuk, *A compact space having the cardinality of the continuum with no convergent sequences*, Math. Proc. Camb. Phil. Soc. 81(1977), 177-181.

[6] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand Reinhold Co., New York 1960.

[7] R. Haydon, *Three examples in the theory of spaces of continuous functions*, Publications du Dept. de Math., Lyon, 9(1972), 99-103, also Comptes Rendus 276 Series A(1973) 685-687.

[8] J.L. Kelley, *General Topology*, D. Van Nostrand, Princeton, New Jersey 1955.

[9] K. Kunen, *Large homogeneous compact spaces, in Open problems in topology*, ed. by Jan van Mill and George M. Reed, 261-270, Elsevier Science Publisher, Amsterdam-New York 1990.

[10] V.I. Malykhin, *$\beta$N is prime*, Bull. Pol. Acad. Sci. 27(1979) 295-297.

[11] B.D. Shapirovski, *Maps onto Tikhonov cubes*, Uspekhi Mat. Nauk. 35:3 (1980) 122-130, Russian Math. Surveys 35:3 (1980) 145-146.

[12] J. van Mill, *An introduction to $\beta$N, in Handbook of Set-Theoretic Topology*, 503-567. ed. by K. Kunen and J. Vaughan, North-Holland, Amsterdam-New York-Oxford 1984.

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