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GENERATING CELLULAR DECOMPOSITIONS OF E^3 AND THE NONEXISTENCE OF CERTAIN FINITE-TO-ONE MAPPINGS

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1. Introduction

Some years ago, Hurewicz constructed monotone mappings m of compacta in E^3 onto given compacta Y. The sets $m^{-1}m(x)$ for $x \in E^3$ are, of course, compact and connected, but without various other connectivity properties ($LC^n, lc^n, n-LC$, etc., for $n > 0$). It is difficult to use Hurewicz's technique (and, indeed, generally impossible) to construct cellular mappings from compacta in E^3 onto certain 2-dimensional polyhedra.

We shall give conditions under which Hurewicz's technique can be modified to yield very nice cellular decompositions of E^3 (closed mappings f defined on E^3 with $f^{-1}f(x)$ cellular for each $x \in E^3$). It will be shown that it is impossible to use his technique (in some sense) to obtain certain special cellular decompositions of E^3. A surprising consequence is that certain finite-to-one mapping from the Cantor Space onto any n-cell do not exist.

Some interesting general questions arise.

Suppose that K^n is an n-dimensional polyhedron (more generally, an n-dimensional compactum). When does there exists a metric space (Y, d) and a closed cellular mapping f of E^3 onto Y such that Y contains a homeomorphic copy of K^n? What is the least natural number m such that for each metric space (Y, d) and each closed cellular mapping f of E^3 onto Y, E^m contains a homeomorphic copy of Y? Some related work is contained in [1; 2; 3; 4; 5].

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2. Finite-to-one mappings on the Cantor Space

The existence of certain finite-to-one (continuous) mappings defined on the usual Cantor Space (contained in the closed interval $[0,1]$) imply the existence of very nice cellular decompositions of E^3. Consider the following.

Definition. Suppose that f is a continuous mapping from C (the Cantor Space) into a metric space (X,d). The collection $G_f = \{f^{-1}f(x)|x \in C\}$ is *not properly situated on* C if and only if for some $g \in G_f$ containing points a, b and c with $a < b < c$, there is a sequence $\{g_i\}$ in G_f such that there are points $a_i, c_i \in g_i$ with $a_i < c_i$, $\lim a_i = a$ and $\lim c_i = c$, but there is no sequence $\{b_i\}$ of points $b_i \in g_i$ such that $a_i < b_i < c_i$. Otherwise, G_f is *properly situated on* C. The mapping f is said to be *admissible on* C iff G_f is properly situated on C.

3. Generating cellular decompositions of E^3 with each non-degenerate element being a simple polygonal arc

Theorem 1. *Suppose that K is a compactum and that f is a finite-to-one admissible mapping of C onto K. Then there is a cellular decomposition G of E^3 such that the decomposition space E^3/G contains a homeomorphic copy of K.*

Proof. Let J be the join of the interval $A = [0,1]$ (containing C) with another copy $B = [0,1]$. Suppose that $g \in G_f$ lies in A. Let g' denote the copy of g in B. Write $g = \{p_1, p_2, \cdots, p_n\}$ where $p_i < p_{i+1}$ for $i = 1, 2, \cdots, n-1$. Similarly, write $g' = \{p'_1, p'_2, \cdots, p'_n\}$ where $p'_i < p'_{i+1}$. Now, in the join J construct a simple polygonal arc as follows. Connect p_i to p'_i with a straight line interval with end points p_i and p'_i. Also, connect p_i to p'_{i+1} with a straight line interval with these points as end points for $i = 1, 2, \cdots, n-1$. The union of these straight line intervals in a simple polygonal arc, which we denote gg'.

Let G be the collection of all such gg' for $g \in G_f$ and all singletons x in E^3 not on one of these arcs.

It is easy to see that G is an upper semicontinuous decomposition of E^3, since f is admissible on C. Consequently, the quotient mapping $p : E^3 \to E^3/G$ is a closed cellular mapping with $p^{-1}p(x) \in G$ for each $x \in E^3$. Clearly, E^3/G contains a homeomorphic copy of K.

The dimension of K in the above theorem can not exceed *three*, since

Kozlowski and Walsh [7] have proved that cell-like mappings on 3-manifolds do not raise dimension.

It has been proved by Flores [5] that the complex K^n consisting of all faces of dimension less than or equal to n of a $(2n+2)$-simplex cannot be embedded in E^{2n}. Thus, the following corollary would reduce an oustanding question to one of finding an admissible finite-to-one mapping of the Cantor Space onto the 2-skeleton K^2 of a 6-simplex σ^6.

Corollary. *If there is a finite-to-one admissible mapping f of C onto the 2-skeleton K^2 of a 6-simplex σ^6, then there is a cellular upper semicontinuous decomposition G of E^3 such that $E^3/G \times E^1$ is not homeomorphic to E^4.*

Proof. Construct G as in Theorem 1. If $E^3/G \times E^1$ is homeomorphic to E^4, then K^2 embeds in E^4 contrary to the work of Flores. Consequently, $E^3/G \times E^1$ is not homeomorphic to E^4.

A result of Daverman and Preston [4] states that if G is a cell-like usc (upper semicontinuous) decomposition of E^3 such that the image under the projection mapping $p : E^3 \to E^3/G$ of the set $N_G = \{x | p^{-1}p(x) \neq x\}$ has dim ≤ 1, then $E^3/G \times E^1$ is homeomorphic to E^4. There are related results. However, the question of whether or not $E^3/G \times E^1$ is homeomorphic to E^4 for *each* cell-like (or cellular) decomposition G of E^3 remains unanswered.

The following theorem states that any one-dimensional compactum has a homeomorphic copy in some nice cellular decomposition of E^3.

Theorem 2. *Suppose that K is a one-dimensional compactum. Then there is an usc decomposition G of E^3 such that each non-degenerate element of G is a simple polygonal arc and E^3/G contains a homeomorphic copy of K.*

Proof. There is a mapping f of C onto K such that $|f^{-1}f(x)| \leq 2$ for each $x \in C$. Note that f must be admissible. Construct G as in the proof of Theorem 1.

Suppose that K is an n-dimensional compactum. Then there is a mapping f of C onto K such that $|f^{-1}f(x)| \leq n+1$ for each $x \in C$ and the collection $G_f^{n+1} = \{f^{-1}f(x) | f^{-1}f(x)$ is exactly $n+1$ points$\}$ is countable. This follows easily from the covering dimension of K and the 0-dimensionality of C. However, G_f^{n+1} *may not be a null collection.* Recall that G_f^{n+1} is null iff for each $\varepsilon > 0$, at most a finite number of the elements of G_f^{n+1} have diam $> \varepsilon$. We shall show that no such mappings f exist in some situations with G_f^3 being a null collection. First, consider the following theorem.

Theorem 3. *Suppose that K is a compactum and that f is a mapping of C onto K which is at most three-to-one. If G_f^3 is a null collection then there is a cellular usc decomposition G of E^3 such that E^3/G contains a homeomorphic copy of K.*

Proof. Write $G_f^3 = \{T_i\}$. Thus, diam $T_i \to 0$. For each i, let $T_i = \{a_i, b_i, c_i\}$, where $a_i < b_i < c_i$. Let $H_j = \{f^{-1}f(x) | f^{-1}f(x)$ is exactly j points $\}$. Then $H_3 = G_f^3$.

For each i, construct a disk $D(T_i)$ bounded by three semi-circles with end points a_i and b_i, a_i and c_i, and b_i and c_i, all lying in the half-plane containing the x-axis making an angle of a_i with the xy-plane.

There exist pairwise disjoint intervals $I(a_1), I(b_1)$ and $I(c_1)$ in $[0,1]$ such that:

(1) $a_1 \in I(a_1), b_1 \in I(b_1)$ and $c_1 \in I(c_1)$;

(2) $[I(a_1) \cup I(b_1) \cup I(c_1)] \cap C = A_1$ is an open and closed subset of C; and

(3) for $i > 1$, $T_i \cap I(p) \neq \emptyset$ for at most one of $p = a_1, b_1$ or c_1.

Let $S_1 = \{g | g \in H_2 \cup H_3, A_1 \supset g$, and g is not contained entirely in one of $I(a_1), I(b_1)$ and $I(c_1)\}$. Note that S_1^* is closed (where M^* denotes the union of the elements of the collection M). Also, $T_1 \in S_1$ and $T_i \notin S_1$ for $i > 1$.

Let h_1 denote an order-preserving homeomorphism of $I(b_1) \cap C$ into $I(a_1)$ such that $h_1(b_1) = a_1$ and $h_1(x) \notin I(a_1) \cap C$ for $x \in I(b_1) \cap (C - b_1)$.

Now, if $g \in S_1$ and $g = \{x, y\}$, $x < y$, then construct a semi-circle $C(g)$ with end points x and y lying in the half-plane containing the x-axis and making either an angle of x with the xy-plane if $x \in I(a_1)$ or an angle of $h_1(x)$ if $x \in I(b_1)$.

Consider $T_2 = \{a_2, b_2, c_2\}$. Construct pairwise disjoint intervals $I(a_2), I(b_2)$ and $I(c_2)$ in $[0,1]$ such that:

(1) $a_2 \in I(a_2), b_2 \in I(b_2)$ and $c_2 \in I(c_2)$;

(2) $[I(a_2) \cup I(b_2) \cup I(c_2)] \cap C = A_2$ is an open and closed subset of C;

(3) $A_2 \cap S_1^* = \emptyset$;

(4) each interval has length $< 1/2^2$; and

(5) if $i > 2$, then $T_i \cap I(p) \neq \emptyset$ for at most one of $p = a_2, b_2$ or c_2.

Let h_2 denote an order-preserving homeomorphsim of $I(b_2) \cap C$ into $I(a_2)$ such that $h_2(b_2) = a_2$, $h_2(x) \notin I(a_2) \cap C$ for $x \in I(b_2) \cap (C - b_2)$, and $h_2(x) \neq h_1(y)$ for any x and y.

Let $S_2 = \{g|g \in H_2 \cup H_3,\ A_2 \supset g$, and g is not contained entirely in one of $I(a_2), I(b_2)$ and $I(c_2)\}$. Note that S_2^* is closed. Also, $T_2 \in S_2$.

If $g \in S_2$ and $g = \{x, y\}$, $x < y$, then construct a semi-circle $C(g)$ with end points x and y lying in the half-plane containing the x-axis and making either an angle of x with the xy-plane if $x \in I(a_2)$ or an angle of $h_2(x)$ if $x \in I(b_2)$.

Continuing in this manner, we construct, for each i, pairwise disjoint intervals $I(a_{i+1}), I(b_{i+1})$ and $I(c_{i+1})$ in $[0,1]$ such that:

(1) $a_{i+1} \in I(a_{i+1})$, $b_{i+1} \in I(b_{i+1})$ and $c_{i+1} \in I(c_{i+1})$;

(2) $[I(a_{i+1}) \cup I(b_{i+1}) \cup I(c_{i+1})] \cap C = A_{i+1}$ is an open and closed subset of C;

(3) $A_{i+1} \cap [\cup_{k=1}^{i} S_k^*] = \emptyset$, where $S_k = \{g|g \in H_2 \cup H_3, A_k \supset g$, and g is not contained entirely in one of $I(a_k), I(b_k)$ and $I(c_k)\}$;

(4) each interval has length $< 1/2^{i+1}$; and

(5) if $k > i+1$, then $T_k \cap I(p) \neq \emptyset$ for at most one of $p = a_{i+1}$, b_{i+1} or c_{i+1}.

For each i, there is an order-preserving homeomorphism h_i of $I(b_i) \cap C$ into $I(a_i)$ such that:

(1) $h_i(b_i) = a_i$,

(2) $h_i(x) \notin I(a_i) \cap C$ for $x \in I(b_i) \cap (C - b_i)$, and

(3) $h_i(x) \neq h_j(y)$ for any x and y for $i \neq j$.

Note that S_i^* is closed and $T_i \in S_i$.

If $g \in S_i$ and $g = \{x, y\}$, $x < y$, then construct a semi-circle $C(g)$ with end points x and y lying in the half-plane containing the x-axis and making either an angle of x with the xy-plane if $x \in I(a_i)$ or an angle of $h_i(x)$ if $x \in I(b_i)$.

If $\{x, y\} = g \in H_2$ where $g \notin S_i$ for any i, then construct a semi-circle $C(x, y)$ in the half-plane containing the x-axis and making an angle of $-x$ with the xy-plane. Thus, $C(x, y)$ lies *below* the xy-plane whereas the other disks and semi-circles constructed thus far lie above the xy-plane.

Let H denote the collection of the various disks and semi-circles constructed in this manner. Let G denote the collection consisting of the elements of H together with the singletons in E^3 which do not belong to an element of H. It follows that G is a cellular (point-like) usc decomposition of E^3. Furthermore, the decomposition space E^3/G contains a homeomorphic copy of K.

The proof of Theorem 3 can be easily adapted to prove the following theorem.

Theorem 4. *Suppose that f is an at most three-to-one mapping of the Cantor Space C onto a compactum K. Furthermore, $G_f^3 = \{\{a_k, b_k, c_k\} | k = 1, 2, 3, \cdots\}$ is countable, and, for each i and each $\varepsilon > 0$, there exist pairwise disjoint closed intervals $I(a_i), I(b_i)$ and $I(c_i)$ containing a_i, b_i and c_i, respectively, such that:*

(1) $[I(a_i) \cup I(b_i) \cup I(c_i)] \cap C = A_i$ *is open and closed;*

(2) diam $I(p) < \varepsilon$ *for* $p = a_i, b_i$ *and* c_i*; and*

(3) *if* $T_j = \{a_j, b_j, c_j\}$ *meets two of these intervals, then* $A_i \supset T_j$.

Then there is a cellular usc decomposition G of E^3 such that E^3/G contains a homeomorphic copy of K.

4. The nonexistence of a three-to-one mapping of C onto a 2-simplex

If there is a three-to-one (continuous) mapping f of C onto a 2-simplex σ^2 such that G_f^3 is a null collection, then one can modify f so that for each y in a face (1-simplex) of σ^2, $f^{-1}(y)$ is at most two points. It is not difficult to see that this would imply the existence of a mapping g onto the projective plane P such that g is at most three-to-one and G_g^3 is a null collection. We now state the following:

Theorem 5. *There is no mapping f of C (the Cantor Space) onto a 2-simplex σ^2 such that f is at most three-to-one and G_f^3 is a null collection.*

Proof. If the theorem is false, then construct (as indicated above) an at most three-to-one mapping g of C onto the projective plane P such that G_g^3 is a null collection. By Theorem 3, there is a closed mapping Ψ of E^3 onto a metric space (Y, d) where Y contains a homeomorphic copy Q of P.

Let $X = \Psi^{-1}(Q)$, a compactum in E^3, and let $\Theta = \Psi|_X$. Using the integers Z for coefficients, the second cohomology group $H^2(Q)$ of Q is isomorphic to Z_2 (integers mod 2). By a Vietoris-Begle theorem [8], Θ induces an isomorphism on Čech cohomology. Hence, $\check{H}(X)$ is isomorphic to Z_2. By Alexander duality [8], $\check{H}(X) \simeq \check{H}_0(E^3 - X) \simeq Z_2$. However, this is impossible. Hence, the theorem is proved.

We are indebted to Ross Geoghegan for suggesting this proof.

For similar reasons there is no admissible mapping f of C onto the projective plane P.

Question. For what compacta K are there admissible mappings f from the Cantor Space C onto K?

5. Certain k-to-1 mappings, $k \geq 3$, raise dimension by at most $k-2$.

A well-known theorem of Hurewicz [6, p.91] states that if f is a closed mapping of a separable metric space (X, d) onto a metric space (Y, ρ) and $\dim X - \dim Y = k > 0$, then there is a point $y \in Y$ such that $\dim f^{-1}(y) \geq k$. A kind of dual of this theorem states that if $\dim Y - \dim X = k > 0$, then there is at least one point $y \in Y$ such that $|f^{-1}(y)| \geq k+1$.

It is also well-known that if Y is a k-dimensional compactum, then there is a mapping f of the Cantor Space C onto Y such that for each $y \in Y$, $|f^{-1}(y)| \leq k+1$. The question of the dimensionality of the image Y of C under a continuous mapping f such that for each $y \in Y$, $|f^{-1}(y)| \leq k$ seems not to have been answered (and, maybe, not asked).

Notation. Suppose that f is a finite-to-one mapping of X onto Y. Let $N(x, f) = |f^{-1}f(x)|$ for $x \in X$. Let $N(f) = \sup\{N(x, f) | x \in X\}$. Let $H_i(f) = \{y | y \in Y \text{ and } N(x, f) \geq i \text{ for } x \in f^{-1}(y)\}$.

In a letter, John J. Walsh gave a short easy proof that if f is a continuous mapping of C onto a compactum Y such that $N(f) = 3$, $H_3(f) = \{y_1, y_2, \cdots,\}$ is countable and $\{f^{-1}(y_i)\}$ is a null sequence, then $\dim Y \leq 1$. We were able to find a proof of a much more general result which is given below. It is an interesting result which seems to have been overlooked by dimension theorists.

Theorem 6. *Suppose that each of (X, d) and (Y, ρ) is a separable metric space and that f is a continuous mapping of X onto Y such that $f^{-1}f(x)$ is finite for each $x \in X$. If $k \geq 3$, $H_k(f) = \{y_1, y_2, \cdots\}$ and $\{f^{-1}(y_i)\}$ is a null sequence, then* $\dim Y \leq \dim X + k - 2$.

Proof. The sequence $\{f^{-1}(y_i)\}$ is a null sequence. Thus, there are closed neighborhoods V_i of y_i such that $f^{-1}(V_i) = U_1^i \cup U_2^i \cup \cdots \cup U_{j_i}^i$ where each U_t^i is closed, $U_s^i \cap U_t^i = \emptyset$ if $s \neq t$, each U_t^i contains exactly one point of $f^{-1}(y_i)$, and, if $y \neq y_i$ and $f^{-1}(y)$ meets at least two of the sets U_t^i, then $|f^{-1}(y)| \leq k-1$. Note that the set $K_i = \{y | f^{-1}(y)$ meets at least two of the sets $U_t^i\}$ is closed. Let $L_i = \{y || f^{-1}(y)| > 1$ and $f^{-1}(y)$ is a subset of some $U_t^i\}$. Now, for each natural number i and each natural number n, let $L_i(n) = \{y|y \in L_i$ and diam $f^{-1}(y) \geq 1/n\}$. Thus, $L_i(n)$ is closed for each i and n. The set $W_i(n) = int(V_i - L_i(n))$ is an open set.

Since $g = f|_{f^{-1}(K_i)}$ is a closed mapping and $N(x,g) \leq k-1$, except for x such that $f(x) = y_i$, it follows that $\dim K_i \leq \dim X + k - 2$. (That $k \geq 3$ is needed here.) For $y \in W_i(n) - y_i, |f^{-1}(y)| \leq k-1$. Thus, $W = \cup_{i=1}^{\infty}(\cup_{n=1}^{\infty} W_i(n))$ is open, and $\dim W \leq \dim X + k - 2$, since $f|_{f^{-1}(W)}$ is closed and $W \supset H_k(f)$. If $y \notin W$, then $|f^{-1}(y)| \leq k-1$. Thus, $f|_{(X - f^{-1}(W))}$ is an at most $(k-1)$-to-one closed mapping on a closed set. This implies that $\dim(Y - W) \leq \dim X + k - 2$. Since W is an F_σ-set and $Y - W$ is closed, $\dim Y \leq \dim X + k - 2$. The theorem is proved.

Remarks. There is an obvious two-to-one mapping f of the Cantor Space onto the closed interval $[0,1]$ such that $H_2(f)$ is a countable set $\{y_1, y_2, \cdots\}$ and $\{f^{-1}(y_i)\}$ is a null sequence. Thus, the assumption that $k \geq 3$ is necessary. However, there is no two-to-one mapping of the Cantor Space onto the plane one-dimensional Sierpinski curve such that $H_2(f)$ is *countable.*

It would be interesting to classify those one-dimensional images of the Cantor Space under two-to-one mappings f such that $H_2(f)$ is a countable set $\{y_1, y_2, \cdots\}$ and $\{f^{-1}(y_i)\}$ is a null sequence. More generally, we may attempt to classify those spaces Y such that there is a (continuous) finite-to-one mapping f of X onto Y such that $k \geq 3$, $H_k(f) = \{y_1, y_2, \cdots\}$ is countable and $\{f^{-1}(y_i)\}$ is a null sequence. In some sense, these spaces Y which have $\dim Y \leq \dim X + k - 2 = m$ are "thinner" than some spaces of dimension m.

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