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ON MODULAR PAIRS IN LOOMIS-$*$-SEMIGROUP

S. H. Raza

The purpose of this paper is to introduce the concept of modularity in loomis-$*$-semigroup. In consideration of this we extend some new results on loomis-$*$-semigroup. D.J. Foulis [1], proved that weak loomis-$*$-semigroup is a complete Baer-$*$-semigroup and a loomis-$*$-semigroup is a weak Loomis-$*$-semigroup satisfying the (EP) property. Particularly we are interested in the use of loomis-$*$-semigroup as Baer-$*$-semigroup. By virtue of this we proved different new results. In this note we shall assume that the reader is familiar with the terminology and notations of the quoted papers. The question arises: What connection exists if the loomis-$*$-semigroup posses modular property. We shall prove that if in a loomis-$*$-semigroup modular property holds with $a : e \sim_* f$ then $(ga)''a^* = (g'a)'a^*$ and another important theorem. In a loomis-$*$-semigroup if $(e, f)M$, for every $e, f \in P'(S)$ and e is $*$-equivalent to a subelement of f if and only if f is $*$-equivalent to a subelement of e, then

$$a_1 \wedge b_1 = ((ea_1^*)'a_1)'' \wedge ((fb_1^*)'b_1)'' \text{ where } a_1 = f'n \text{ and } b_1 = e'm.$$

Apart from this we also proved other interesting theorems.

Definitions. A $*$-semigroup S is a semigroup with a mapping $* : S \to S$ such that (i) $(xy)^* = y^*x^*$ (ii) $x^{**} = x$ for all $x, y \in S$. An element $e \in S$ is called projection when $e = e^* = e^2$. A projection e is closed when $e = e'' \in S$. The set of projections and closed projections are denoted by $P(S)$ and $P'(S)$ respectively. A Baer $*$-semigroup is an involution semigroup with when for each element $a \in S$ there exists a projection e of S such that $\{x \in S; ax = 0\} = e'S$. We say that e and f are $*$ equivalent

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and denoted by $x : e \sim_* f$ where x is partially unitary, when $x = exf$, $x^*x = f$ and $xx^* = e$. If M is any non-empty subset of S, then $Z(M)$ is the centralizer of M if $Z(M) = \{s \in S : sx = xs \text{ for all } x \in M\}$. We define $zz(M) = Z(Z(M))$. The involution semigroup S has the property (EP) if given any non-zero element $x \in S$, there exists an element $y \in S$ such that $y = y^* \in zz(x^*x)$ and x^*xZ is a non-zero closed projection, then we shall say that S has property (WEP). An involution semigroup S satisfying $*$-cancellation law and (WEP) property in which all projections are closed and if $\{a_{\mathcal{L}}\}$ is orthogonal family of partially unitary elements of S, then $\text{sum}_{\mathcal{L}}\, a_{\mathcal{L}}$ is not empty. A loomis $*$-semigroup is a weak Loomis-$*$-semigroup satisfying the (EP) property.

Theorem 1. *Let S be a Loomis-$*$-semigroup, $a : e \sim_* f$ and $(e, f)M$ $e, f \in P'(S)$. Then for $g \in P'(S)$*

$$(ga)''a^* = (g'a)'a^*.$$

Proof. Since $(ga)'' = [(g' \wedge e)a]' \wedge f$ by lemma 18[2]. So $(ga)' \wedge f = ([(g' \wedge e)a]'' \vee f') \wedge f = [(g' \wedge e)a]''$, since $[(g' \wedge e)a]'' \leq a'' = (eaf)'' \leq f$ by 37.5.3 [1] and $(f', f)M$. Now by 37.7 [1], $(ga)' \wedge f = [(g'e)a]'' = [g'(ea)]'' = (g'a)''$, since $aa^* = (ea)a^* = (ea)(ea)^*$ so $a = ea$ by $*$-cancellation law. Hence from $[(ga)' \wedge f] \vee f' = (g'a)'' \vee f'$ we have $[(ga)'' \vee f'] \wedge f = (g'a) \wedge f$, which further gives $(ga)'' = (g'a)' \wedge f$, since $(ga)'' \leq a'' = (eaf)'' \leq f$ by 37.5.3 [1] and $(f', f)M$. As $(g'a)'' \leq f$, so by 37.5[1] $f' \leq (g'a)'$ which implies $f'(g'a)' = (g'a)'f' = f'$ and $f(g'a)' = (g'a)'f = f \wedge (g'a)'$ by 37.5 ad 37.7[1]. Now on multiplying $(ga)'' = f(g'a)'$ by a on the left, we get $a(ga)'' = af(g'a)'$ and so $a(ga)'' = a(g'a)'$; since $a = eaf$ and $ea = a$. On taking involution on both sides we get $(ga)''a^* = (g'a)'a^*$.

Theorem 2. *Let S be a Loomis-$*$-semigroup, $a : e \sim_* f$ and $(e, f)M$ for every $e, f \in P'(S)$. Then*

$$a = eba \text{ where } b = \{[e(f'a')']' \wedge f\}'.$$

Proof. From 37.5[1] $a'' = (eaf)'' \leq f$, $f' \leq a'$. Therefore $(f'a')' = f, [e(f'a')']' = [ef]'$. Hence $[e(f'a')']' \wedge f = (ef)' \wedge f = \{(e' \wedge f) \vee f'\} \wedge f = (e' \wedge f)$ from 37.10 [1], since $(f', f)M$. Now $\{[e(f'a')']' \wedge f\}' = e \vee f'$ gives $\{[e(f'a')']' \wedge f\}'e = (e \vee f')e = e$, since $e \leq e \vee f'$. Let $\{[e(f'a')'] \wedge f\}' = b$. So $a^*be = a^*e = a^*$, since $aa^* = (ea)a^* = (ea)(ea)^*$ and so $ea = a$ by $*$-cancellation. Similarly $af = a$. Now $eb^*a = a = ea$ and so $eb^*af = eaf = a$. Therefore $eba = a$.

Theorem 3. *Let e is $*$-equivalent to a subelement of f if and only if f is $*$-equivalent to a subelement of e and $(e,f)M$ for every $e,f \in P'(S)$, then*

$$a_1 \wedge b_1 = ((ea_1^*)'a_1)'' \wedge ((fb_1^*)'b_1)''$$

where $a_1 = f'n$ and $b = e'm$.

Proof. As $a : e \sim_* f_1 \leq f$, $b : f \sim_* e_1$, so that exists projections $m, n \in P'(S)$ with $m \leq e$, $n \leq f$; $n = (ma)' \wedge f$ and $m = (nb)' \wedge e$ from Theorem 21[2] since $(ma)'' = [(m' \wedge e)a]' \wedge f$, $(nb)'' = [(n' \wedge f)b]' \wedge e_1$ from theorem 18[2] we have $(ma)'' \leq f_1 \leq f$, $(nb)'' \leq e_1 \leq e$, $(ma)'' \vee f = f$ and $(nb)'' \vee e = e$. Also $m \leq e$, $n \leq f$ gives $me = em = m$, $nf = fn = n$ and so $me' = e'm = m\{(nb)' \wedge e'\}$ and $nf' = f'n = n\{(ma)' \wedge f'\}$ from 37.5[1]. Now

$$\begin{aligned} e' \wedge nf' &= e' \wedge f'n = n\{(ma)' \wedge f'\} \wedge e' \\ &= ((f'n)'e')'e' = ((ea_1^*)'a_1)'' \text{ from 37.8[1]} \\ &= ((ea_1^*)'a_1)'' \text{ where } a_1 = f'n, \text{ from 38.2[1]}. \end{aligned}$$

Similarly, $e'm \wedge f' = m\{(nb)' \wedge e'\} \wedge f' = ((fb_1^*)'b_1)''$ where $b_1 = e'm$. Therefore $\{(me') \wedge e'\} \wedge \{(nf') \wedge f'\} = ((ea_1^*)'a_1)'' \wedge ((fb_1^*)'b_1)''$. So finally we get $a_1 \wedge b_1 = ((ea_1^*)'a_1)'' \wedge ((fb_1^*)'b_1)''$, since $(me')'' \leq e'$, $(nf')'' \leq f'$ from 37.5[1].

Theorem 4. *Let $\{f_\alpha\}$ be an orthogonal family of projections in S and suppose that $a : e \sim_* V_\alpha f_\alpha$ where e is a projection in S for every $e,f \in P'(S)$, then*

$$a_\alpha'' V[a_\alpha'(a_\alpha' \wedge (g'a_\alpha)')a_\alpha^*\}'] = f_\alpha, g \in P'(S).$$

Proof. Now there exists a decomposition $e = V_\alpha e_\alpha$ of e into orthogonal projections $\{f_\alpha\}$ such that $a_\alpha : e_\alpha \sim_* f_\alpha$ by Theorem 19[2] which gives $a_\alpha a_\alpha^* = e_\alpha$ and $a_\alpha^* a_\alpha = f_\alpha$. Now a_α is a non-zero element then by lemma 3[2], there exists a nonzero closed projection $g \in ZZ(a_\alpha^* a_\alpha)$ and an element $y = y^* \in ZZ(a_\alpha^* a_\alpha)$ such that $a_\alpha^* a_\alpha y = g$ and $y = gyg$. We have $f_\alpha y = g$, because $a_\alpha^* a_\alpha y = g$ and $a_\alpha^* a_\alpha = f_\alpha$. Since $a_\alpha a_\alpha^* = (ea_\alpha)a_\alpha^* = (ea_\alpha)(ea_\alpha)^*$, we have by $*$-cancellation law $a_\alpha = ea_\alpha$.

Now $[(g' \wedge e_\alpha)a_\alpha]'' \leq a_\alpha''$ and $a_\alpha' \leq [(g' \wedge e_\alpha)a_\alpha]'$ by 37.5[1]. We get $a_\alpha' \wedge [(g' \wedge a_\alpha)a_\alpha]' = a_\alpha'$. So $a_\alpha' \wedge [(g' \wedge e_\alpha)a_\alpha]' \wedge f_\alpha = a_\alpha' \wedge f_\alpha$. Now from lemma 18[2] and (37.7) [1] $a_\alpha' \wedge [(g' \wedge e_\alpha)a_\alpha]' \wedge f_\alpha = a_\alpha'(ga_\alpha)''$. As

$a_\alpha = (e_\alpha a_\alpha f_\alpha)$, so $a''_\alpha = (e_\alpha a_\alpha f_\alpha)'' \leq f_\alpha$ by 37.5[1]. By 37.5[1] $f'_\alpha \leq a'_\alpha$. Hence we have $f'_\alpha a'_\alpha = a'_\alpha f'_\alpha = f'_\alpha$. Further $f_\alpha a'_\alpha = a'_\alpha f_\alpha = a'_\alpha \wedge f_\alpha$ by 37.7[1]. Now $a'_\alpha f_\alpha = 0$, since $f_\alpha = a^*_\alpha a_\alpha$, hence $a'_\alpha f_\alpha = (a'_\alpha a^*_\alpha)a_\alpha = 0$ by 37.5[1]. From (i) we get $a'_\alpha(ga_\alpha)'' = 0$. So $a'_\alpha(ga_\alpha)'' a^*_\alpha = 0$. By Theorem 1 $a'_\alpha(g'a_\alpha)'a^*_\alpha = 0$. Hence $[a'_\alpha \wedge (g'a_\alpha)']a^*_\alpha = a'_\alpha f_\alpha$, by Theorem 37.7[1] which implies $a'_\alpha([a'_\alpha \wedge (g'a_\alpha)']a^*_\alpha)' = a'_\alpha(f_\alpha a'_\alpha)'$. Also $(f_\alpha a'_\alpha)'' \leq a'_\alpha$ by 37.5[1] which gives $a'_\alpha(f_\alpha a'_\alpha)'' = (f_\alpha a'_\alpha)'' = ((f_\alpha a'_\alpha)'' a'_\alpha = (f_\alpha a_\alpha)''$. By (37.5.8), (37.7) [1], $(f_\alpha a'_\alpha)'a'_\alpha = a'_\alpha(f_\alpha a'_\alpha)' = a'_\alpha \wedge (f_\alpha a'_\alpha)$, $a'_\alpha(a'_\alpha \wedge (g'a_\alpha)'a^*_\alpha)'$ $= [a'_\alpha \wedge (f_\alpha a'_\alpha)''] = a'_\alpha \wedge f_\alpha$. So $(ea_\alpha)'' \vee [a'_\alpha(a'_\alpha \wedge (g'a_\alpha)'a^*_\alpha]' = a''_\alpha \vee [a'_\alpha \wedge f_\alpha] = f_\alpha$, since $a''_\alpha = f_\alpha$ and $(e,f)M \wedge e, f \in P'(S)$.

References

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Department of Mathematics, Swami Shridhananda College, Alipur, Delhi University, Delhi, India.