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ON THE CONSTRUCTION OF INDEPENDENT SECTIONS OF $T(S^{2m+1} \times S^n)$

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1. Introduction

Let S^n denote the unit n-sphere in the Euclidean $(n+1)$-space R^{n+1} and $T(M)$ the tangent bundle of a smooth manifold M.

Utilizing productivity of Euler characteristic classes and the fact that product of two stably trivial bundles is trivial if one of them has at least one independent section (cf. [2], [3], [4] or [5]), we see that vector fields problem of $S^{2m+1} \times S^n$ turns out to be almost obvious compared with Adams' work on spheres ([1]).

The purpose of this paper is to show how to construct $2m+n+1$ independent sections of $T(S^{2m+1} \times S^n)$ with a given independent section of $T(S^{2m+1})$.

The construction is carred out as follows. Using a single section S of $T(S^{2m+1})$, we construct $n+1$ independent sections $U_1, \cdots, U_{n+1}$ of $\theta(S) \oplus T_n$, where $\theta(S)$ is the trivial bundle associated with S, T_n is the pull-back bundle of $T(S^n)$ under the canonical projection, and $\oplus$ denotes the Whitney sum of vector bundles. Next step is to construct $2m+2$ independent sections $V_1, \cdots, V_{2m+2}$ of $T'_{2m} \oplus \theta(U_1) \oplus \theta(U_2)$, where $T'_{2m} = T_{2m+1}/\theta(S)$, the quotient bundle of T_{2m+1} by $\theta(S)$, and $\theta(U_i)$ is the trivial bundle associated with U_i for $i = 1, 2$. Then $\{U_i, V_j | 3 \le i \le n+1, 1 \le j \le 2m+2\}$ is the desired set of $2m+n+1$ independent sections of $T(S^{2m+1} \times S^n)$.

2. Description of Subbundles of $T(S^{2m+1} \times S^n)$

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Let T denote the tangent bundle $T(S^{2m+1} \times S^n)$ and let T_{2m+1}, T_n denote the pull-back bundles $\pi_1^*T(S^{2m+1}), \pi_2^*T(S^n)$, respectively, under the canonical projections $\pi_1 : S^{2m+1} \times S^n \to S^{2m+1}$, $\pi_2 : S^{2m+1} \times S^n \to S^n$. Then T, T_{2m+1}, and T_n can be described as follows:

$$T = \{(p, q, u, v) \in S^{2m+1} \times S^n \times R^{2m+2} \times R^{n+1} | < p, u >= 0 =< q, v >\},$$

$$T_{2m+1} = \{(p, q, u, 0) | < p, u >= 0\},$$
$$T_n = \{(p, q, 0, v) | < q, v >= 0\},$$

where $<,>$ denote the standard inner product on R^k. Observe that $T = T(S^{2m+1}) \times T(S^n) = T_{2m+1} \oplus T_n$.

Let $S : S^{2m+1} \to T(S^{2m+1})$ be the independent section defined by

$$\begin{aligned} S(p) &= S(x_1, \cdots, x_{2m+2}) = (-x_2, x_1, -x_4, x_3, \cdots, -x_{2m+2}, x_{2m+1}) \\ &\in T(S^{2m+1})_p, \end{aligned}$$

where $T(S^{2m+1})_p$ is the tangent plane at $p \in S^{2m+1}$. Then the trivial subbundle $\theta(S)$ of T associated with S and $T'_{2m} = T_{2m+1}/\theta(S)$ have the following total spaces:

$$\theta(S) = \{(p, q, \lambda S(p), 0) | \lambda \in R\}$$

$$T'_{2m} = \{(p, q, u, 0) | < p, u >=< S(p), u >= 0\}.$$

Lemma 1. *The bundle* $\theta(S) \oplus T_n = \{(p, q, \lambda S(p), v) | \lambda \in R, < q, v >= 0\}$ *is trivial with the trivialization* α *defined by*

$$\alpha(p, q, \lambda S(p), v) = (p, q, v + \lambda q).$$

Proof. $\alpha : \theta(S) \oplus T_n \to (S^{2m+1} \times S^n) \times R^{n+1}$ is obviously a bundle map with inverse α^{-1} defined by

$$\alpha^{-1}(p, q, z) = (p, q, < q, z > S(p), z - < q, z > q).$$

Let $\{E_i | 1 \leq i \leq n+1\}$ be a fixed orthonormal basis of R^{n+1}. Define a map $U_i : S^{2m+1} \times S^n \to \theta(S) \oplus T_n$ by

$$U_i(p, q) = (p, q, < E_i, q > S(p), E_i - < E_i, q > q)$$

for each $1 \leq i \leq n+1$. Then $\{U_i | 1 \leq i \leq n+1\}$ is clearly a set of independent sections of $\theta(S) \oplus T_n$. Moreover, the trivial bundle $\theta(U_i)$ associated with the section U_i is described by

$$\theta(U_i) = \{(p, q, \lambda < E_i, q > S(p), \lambda E_i - \lambda < E_i, q > q) | \lambda \in R\}.$$

Then the bundle T is isomorphic to $T'_{2m} \oplus (\oplus_{1 \leq i \leq n+1} \theta(U_i))$ and the total space of the subbundle $T'_{2m} \oplus \theta(U_1) \oplus \theta(U_2)$ of T is described by

$$\{(p, q, u + < \lambda_1 E_1 + \lambda_2 E_2, q > S(p), \lambda_1 E_1 + \lambda_2 E_2 - < \lambda_1 E_1 + \lambda_2 E_2, q > q) |$$

$$< u, p >= 0 =< u, S(p) >, \lambda_1, \lambda_2 \in R, u \in R^{2m+2}\}.$$

Lemma 2. *The bundle* $T'_{2m} \oplus \theta(U_1) \oplus \theta(U_2)$ *is trivial with the trivialization* β *defined by*

$$\beta(p, q, u + < \lambda_1 E_1 + \lambda_2 E_2, q > S(p), \lambda_1 E_1 + \lambda_2 E_2 - < \lambda_1 E_1 + \lambda_2 E_2, q > q)$$

$$= (p, q, u + \lambda_1 p + \lambda_2 S(p)).$$

Proof. It is easy to see that $\beta : T'_{2m} \oplus \theta(U_1) \oplus \theta(U_2) \to (S^{2m+1} \times S^n) \times R^{2m+2}$ is a bundle map with inverse β^{-1} defined by

$$\beta^{-1}(p, q, x) = (p, q, \rho_p(x) + << x, p > E_1 + < x, S(p) > E_2, q > S(p),$$

$$< x, p > E_1 + < x, S(p) > E_2 - << x, p > E_1 + < x, S(p) > E_2, q > q),$$

where $\rho_p(x) = x - < x, p > p - < x, S(p) > S(p) \in (T'_{2m})_{(p,q)}$.

3. Main Theorem

For a fixed orthonormal basis $\{F_j | 1 \leq i \leq 2m+2\}$ of R^{2m+2}, define a map $V_j : S^{2m+1} \times S^n \to T' \oplus \theta(U_1) \oplus (U_2)$ by

$$V_j(p, q) = (p, q, \rho_p(F_j) + << F_j, p > E_1 + < F_j, S(p) > E_2, q > S(p),$$

$$< F_j, p > E_1 + < F_j, S(p) > E_2 - << F_j, p > E_1 + < F_j, S(p) > E_2, q > q),$$

for each $1 \leq j \leq 2m+2$. Then $\{V_j | 1 \leq j \leq 2m+2\}$ is a set of linearly independent sections of $T'_{2m} \oplus \theta(U_1) \oplus \theta(U_2)$.

Finally we come up with $2m+n+1$ linearly independent sections

$$\{U_i, V_j | 3 \leq i \leq n+1, 1 \leq j \leq 2m+2\}$$

of $T_{2m+1} \oplus T_n = T$.

Theorem 3. *$T_{2m+1} \oplus T_n$ has $2m+n+1$ linearly independent sections*

$$\{U_i, V_j | 3 \leq i \leq n+1, 1 \leq j \leq 2m+2\}.$$

Proof. It is enough to show that those sections are linearly independent for each $(p,q) \in S^{2m+1} \times S^n$. Let real numbers a_i and b_j be given and let $X = \sum_{3 \leq i \leq n+1} a_i E_i$ and $Y = \sum_{1 \leq j \leq 2m+2} b_j F_j$. Then we see that

$$\sum_{3 \leq i \leq n+1} a_i U_i(p,q) + \sum_{1 \leq j \leq 2m+2} b_j V_j(p,q) = 0$$

if and only if

$$\text{(A)} \qquad < X, q > S(p) + \rho_p(Y) + << Y, p > E_1 + < Y, S(p) > E_2, q > S(p) = 0$$

and

$$\text{(B)} \qquad X - < X, q > q + < Y, p > E_1 + < Y, S(p) > E_2 - << Y, p > E_1 + < Y, S(p) > E_2, q > q = 0.$$

Since $\rho_p(Y)$ is orthonogonal to $S(p)$ for any Y, the equation (A) holds if and only if

$$\text{(C)} \qquad \rho_p(Y) = 0 (\text{i.e., } Y = < Y, p > p + < Y, S(p) > S(p)) \text{ and } < X, q > + << Y, p > E_1 + < Y, S(p) > E_2, q >= 0.$$

Substituting $<< Y, p > E_1 + < Y, S(p) > E_2, q = - < X, q >$ into the equation (B), we get

$$X + < Y, p > E_1 + < Y, S(p) > E_2 = 0.$$

Since $\{E_i | 1 \leq i \leq n+1\}$ is a basis of R^{n+1}, we have

$$< Y, p >=< Y, S(p) >= a_3 = a_4 = \cdots = a_{n+1} = 0.$$

And, from (C), $Y = \sum_{1 \leq j \leq 2m+2} b_j F_j = 0$; hence $b_1 = \cdots = b_{2m+2} = 0$.

References

[1] J. F. Adams, *Vector Fields on Spheres*, Annals of Mathematics Vol. 75, No.3(1962), 603-632.

[2] M. W. Hirsch, Differential Topology, Springer-Verlag, New York, 1976.

[3] D. Husemoller, Fibre Bundles, Springer-Verlag, New York, 1966.

[4] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies No.76, Princeton University Press, Princeton, 1974.

[5] H. Suzuki, *Operations in KO-Theory and Products of Real Projective Spaces*, Memoirs of the Faculty of Science, Kyushu University Ser.A, Vol.18, No.2(1964), 140-153.

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