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SOME ELEMENTARY COMMUTATIVITY THEOREMS FOR ASSOCIATIVE RINGS

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In [3] Johnson, Outcalt, and Yaqub have been proved that "If R is a non-associative ring with identity 1 such that $(xy)^2 = x^2y^2$ for every x and y in R, then R is commutative." Further, Gupta [2] has shown that "If R is a non-associative ring with unity 1 such that $(xy)^2 = (yx)^2$ for every x and y in R and moreover, R is 2-torsion free, then R is commutative."

Throughout $Z(R)$ denotes the center of a ring R. Recently, Yuanchun [4] established the following result "A Baer-semi simple ring R is commutative if and only if $(xy)^2 - xy^2x \in Z(R)$." The existence of a non-commutative ring R with $R^2 \subseteq Z(R)$ rules out the possibility that $(xy)^2 - xy^2x \in Z(R)$ might yield commutativity even R is associative. As an example, we suppose that $A_3 = \{(a_{ij}) | a'_{ij}s$ are integers with $a_{ij} = 0$ for $i \geq j\}$. Then A_3 is a non-commutative nilpotent ring of index 3. Furthermore, $(xy)^2 - xy^2x \in Z(R)$ for each x and y in A_3. This naturally gives rise the following question: "What additional conditions are needed to force the commutativity of a ring R when R is an arbitrary ring?" With this motivation Ashraf, Quadri and Zelinsky [1] proved that "If R is an associative ring with unity 1 such that $(xy)^2 = yx^2y$ for each $x, y \in R$, then R is commutative."

In this note, we prove the following result for associative rings.

Theorem 1. *Let R be an associative ring with unity 1 satisfying $(xy)^2 = xy^2x$ for x and y in R. Then R must be commutative.*

Proof. Replace y by $y+1$ in $(xy)^2 = xy^2x$ to obtain $(x(y+1))^2 = x(y+1)^2x$. Thus

$$x^2y = xyx. \tag{1}$$

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Repeating this argument for $x+1$ in place of x, (1) becomes

$$x^2 + xy = xyx + yx. \tag{2}$$

Combine (2) and (1) to get $xy = yx$. Thus R is commutative.

Similarly, we can prove the following:

Theorem 2. *Let R be an associative ring with unity 1 such that $(xy)^2 = yx^2y$ for all x and y in R. Then R is commutative.*

If we drop the restriction of unity 1 on the hypothesis, R may be badly noncommutative.

Example 1. Let

$$R = \{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} | a, b, c \text{ are integers } \}.$$

A closed look at the above ring shows that $xyz = 0$ for all x, y, z in R. Hence R satisfies the hypothesis of our theorems. But R need not be commutative.

Further, Example 3 of [3] demonstrates that $(xy)^m = x^r y^m x^q$ does not assure commutativity for any choice of r and q such that $r + q = m$ and $m \geq 3$. However we present the following example:

Example 3. Consider

$$R = \{aI + X | X = \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a, b, c, d \in \mathbf{Z}_p\},$$

where p is a prime such that $p|m$ if m is odd or $2p|m$ if m is even and $\mathbf{Z}_p$ is the ring of integers modulo p. Now, $X^3 = 0$, for $m \geq 3$, and

$$(aI + X)^m = a^m I + ma^{m-1}X + m(m-1)/2!a^{m-2}X^2 = a^m I$$

because $m = m(m-1)/2! = 0$ in $\mathbf{Z}_p$ given that $p|m$ or $2p|m(m-1)$. However, R is not commutative.

References

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