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CERTAIN CLASS OF ANALYTIC FUNCTIONS IN THE UNIT DISK

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In this paper we introduce a class of analytic functions satisfying $Re\{(1-\beta)f(z)/z+\beta f'(z)\} > \alpha$ $(0 \leq \alpha < 1,\ 0 \leq \beta \leq 1,\ |z| < 1)$. We study the integral representation formula, coefficient estimates and distortion theorems of such functions. We also consider a subclass of this class of analytic functions.

1. Introduction and Definitions

Let A denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the open unit disk $E = \{z : |z| < 1\}$. Also let $B(\beta, \alpha)$ denote the subclass of A whose members satisfy the inequality

$$Re\{(1-\beta)f(z)/z + \beta f'(z)\} > \alpha(z \in E),$$

where $0 \leq \alpha < 1$, $0 \leq \beta \leq 1$. $B(0,\alpha) = B(\alpha)$ was studied in papers [1]-[5] fully, $B(1,\alpha) = b(\alpha)$ was also studied in papers [5]-[7].

The purpose of this paper is to study some properties of the functions in $B(\beta,\alpha)$, such as integral representation formula, coefficient inequalities and distortion theorems. We also consider a subclass of $B(\beta,\alpha)$ and study the corresponding coefficient inequalities and distortion theorems.

2. The Class $B(\beta,\alpha)$

Lemma 1([9]). *Let $\beta \geq 0$ and $D(z)$ be a starlike function in E. Let $N(z)$ be analytic in E and $N(0) = D(0) = 0$, $N'(0) = D'(0) = 1$. Then $Re\{N(z)/D(z)\} > 0$ for z in E whenever*

$$Re\{(1-\beta)N(z)/D(z) + \beta N'(z)/D'(z)\} > 0$$

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for z in E.

Lemma 2. *Under the same conditions as Lemma 1, we have the conclusion: $Re\{N(z)/D(z)\} > \alpha$ for z in E whenever*

$$Re\{(1-\beta)N(z)/D(z) + \beta N'(z)/D'(z)\} > \alpha$$

for z in E.

Proof. From the condition, we have

$$\frac{1}{1-\alpha} Re\{(1-\beta)(N(z)/D(z) - \alpha) + \beta(N'(z)/D'(z) - \alpha)\} > 0 (z \in E),$$

let $M(z) = (N(z) - \alpha D(z))/(1-\alpha)$, this inequality becomes

$$Re\{(1-\beta)M(z)/D(z) + \beta M'(z)/D'(z)\} > 0, (z \in E).$$

For $M(z)$ and $D(z)$ satisfy the conditions of Lemma 1, we can obtain from Lemma 1 that $Re\{M(z)/D(z)\} > 0 (z \in E)$, this means $Re\{N(z)/D(z)\} > \alpha (z \in E)$.

Theorem 1. *Let $f(z) \in B(\beta, \alpha)$. Then $f(z) \in B(0, \alpha) = B(\alpha)$.*

Proof. From the definition, when $f(z) \in B(\beta, \alpha)$, we have

$$Re\{(1-\beta)f(z)/z + \beta f'(z)\} > \alpha (z \in E). \tag{1}$$

Also for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in E, $f(0) = 0 = f'(0) - 1$, $D(z) = z$ is starlike in E, $D(0) = 0 = D'(0) - 1$, from Lemma 2 and (1) we have

$$Re\{f(z)/z\} > \alpha (z \in E).$$

this means that $f(z) \in B(0, \alpha) = B(\alpha)$.

Theorem 2. *Let $0 \leq \gamma < \beta$. Then $B(\beta, \alpha) \subset B(\gamma, \alpha)$.*

Proof. If $\gamma = 0$, we have proved $B(\beta, \alpha) \subset B(0, \alpha)$ in Theorem 1, so we suppose $\gamma \neq 0$.

When $f(z) \in B(\beta, \alpha)$, we have inequality (1), and from Theorem 1 we also have $Re\{f(z)/z\} > \alpha (z \in E)$, and

$$(1-\gamma)\frac{f(z)}{z} + \gamma f'(z) = \frac{\gamma}{\beta}\{(\frac{\beta}{\gamma} - 1)\frac{f(z)}{z} + (1-\beta)\frac{f(z)}{z} + \beta f'(z)\},$$

thus

$$\begin{aligned}
&Re\{(1-\gamma)\frac{f(z)}{z}+\gamma f'(z)\}\\
&=\frac{\gamma}{\beta}(\frac{\beta}{\gamma}-1)Re\frac{f(z)}{z}+\frac{\gamma}{\beta}Re\{(1-\beta)\frac{f(z)}{z}+\beta f'(z)\}\\
&>\frac{\gamma}{\beta}(\frac{\beta}{\gamma}-1)\alpha+\frac{\gamma}{\beta}\alpha=\alpha \quad (z\in E).
\end{aligned}$$

From the definition we know $f(z)\in B(\gamma,\alpha)$. That means $B(\beta,\alpha)\subset B(\gamma,\alpha)$. The proof is completed.

Theorem 3. *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in B(\beta,\alpha)$. *Then we have sharp coefficient estimates:*

$$|a_n|\le\frac{2(1-\alpha)}{1+(n-1)\beta},(n\ge 2).$$

Proof. Letting

$$(1-\beta)f(z)/z+\beta f'(z)=p(z)=1+\sum_{n=1}^{\infty}c_nz^n, \tag{2}$$

we know $Rep(z)>\alpha(z\in E)$ and

$$|c_n|\le 2(1-\alpha)\quad (n=1,2,\cdots). \tag{3}$$

Substituting the power series expansion of $f(z)$ in (2), we obtain

$$1+\sum_{n=2}^{\infty}[1+(n-1)\beta]a_nz^{n-1}=1+\sum_{n=1}^{\infty}c_nz^n.$$

Comparing the coefficients, we have

$$(1+(n-1)\beta)a_n=c_{n-1}\quad (n\ge 2).$$

Using the estimate (3), we can obtain the inequalities we need to prove. It is easy to know

$$\begin{aligned}
f(z) &= \frac{1}{\beta}z^{1-\frac{1}{\beta}}\int_0^z\zeta^{\frac{1}{\beta}-1}\frac{1+(1-2\alpha)\zeta}{1-\xi}d\zeta\\
&= z+\sum_{n=2}^{\infty}\frac{2(1-\alpha)}{1+(n-1)\beta}z^n
\end{aligned} \tag{4}$$

(where powers are meant as principal values) belongs to $B(\beta,\alpha)$, and it attains the equality in the theorem, so the results are sharp.

Let P denote the class of analytic functions $p(z)=1+\sum_{n=1}^{\infty}c_n z^n$ which satisfies $Rep(z)>0$ $(z\in E)$, i.e., the well-known class of functions with positive real part. We have the following integral representation formula for $f(z)\in B(\beta,\alpha)$.

Theorem 4. *A function $f(z)$ is in $B(\beta,\alpha)$ if and only if there exists $p(z)\in P$ such that*

$$\begin{aligned} f(z) &= \frac{1}{\beta}z^{1-\frac{1}{\beta}}\int_0^z \zeta^{\frac{1}{\beta}-1}[(1-\alpha)p(\zeta)+\alpha]d\zeta \\ &= \frac{z}{\beta}\int_0^1 t^{\frac{1}{\beta}-1}[(1-\alpha)p(zt)+\alpha]dt, \end{aligned} \tag{5}$$

where $\beta\neq 0$. If $\beta=0$, then

$$f(z)=z((1-\alpha)p(z)+\alpha). \tag{6}$$

Powers in (5) are meant as principal values.

Proof. Let $f(z)\in B(\beta,\alpha)$. Then we have inequality (1). So there exists $p(z)\in P$ such that

$$\frac{1}{1-\alpha}\{(1-\beta)f(z)/z+\beta f'(z)-\alpha\}=p(z),$$

that is,

$$(1-\beta)(f(z)-\alpha z)/z+\beta(f'(z)-\alpha)=(1-\alpha)p(z). \tag{7}$$

If $\beta\neq 0$, multiplying both sides of (7) by $(1/\beta)z^{\frac{1}{\beta}-1}$, we obtain

$$[z^{\frac{1}{\beta}-1}(f(z)-\alpha z)]'=\frac{1-\alpha}{\beta}z^{\frac{1}{\beta}-1}p(z).$$

Integrating both sides of this equality from 0 to z, we have

$$\begin{aligned} f(z) &= \alpha z+\frac{1-\alpha}{\beta}z^{1-\frac{1}{\beta}}\int_0^z \zeta^{\frac{1}{\beta}-1}p(\zeta)d\zeta \\ &= \frac{1}{\beta}z^{1-\frac{1}{\beta}}\int_0^z \zeta^{\frac{1}{\beta}-1}[(1-\alpha)p(\zeta)+\alpha]d\zeta \\ &= \frac{z}{\beta}\int_0^1 t^{\frac{1}{\beta}-1}[(1-\alpha)p(zt)+\alpha]dt. \end{aligned} \tag{8}$$

If $\beta = 0$, we can obtain (6) from (7) easily.

Conversely, if $f(z)$ satisfies (5) or (6), then it is easy to see that $f(z) \in B(\beta, \alpha)$. The proof of Theorem 4 is completed.

Theorem 5. *Let $f(z) \in B(\beta, \alpha)$, then for $|z| = r < 1$ we have the following sharp estimates:*

i) *If $\beta \neq 0$,*

$$\frac{r}{\beta}\int_0^1 t^{\frac{1}{\beta}-1}\frac{1-(1-2\alpha)tr}{1+tr}dt \leq |f(z)| \leq \frac{r}{\beta}\int_0^1 t^{\frac{1}{\beta}-1}\frac{1+(1-2\alpha)tr}{1-tr}dt, \quad (9)$$

the function $f(z)$ defined by (4) attains the equality of (9).

ii) *If $\beta = 0$,*

$$r\frac{1-(1-2\alpha)r}{1+r} \leq |f(z)| \leq r\frac{1+(1-2\alpha)r}{1-r}, \quad (10)$$

the function $f(z) = z(1+(1-2\alpha)z)/(1-z)$ attains the equality of (10).

Proof. If $\beta \neq 0$, from the integral representation formula (5) we have

$$|f(z)| \leq \frac{r}{\beta}\int_0^1 t^{\frac{1}{\beta}-1}[(1-\alpha)|p(zt)| + \alpha]dt.$$

Notice that $|p(z)| \leq (1+r)/(1-r)(|z| = r < 1)$. So we obtain the right-side inequality of (9). On the other hand, from (5) we have

$$f(z)/z = \frac{1}{\beta}\int_0^1 t^{\frac{1}{\beta}-1}[(1-\alpha)p(zt) + \alpha]dt,$$

thus

$$Re(f(z)/z) = \frac{1}{\beta}\int_0^1 t^{\frac{1}{\beta}-1}[(1-\alpha)Rep(zt) + \alpha]dt.$$

It follows from $Rep(z) \geq (1-r)/(1+r)$ $(|z| = r < 1)$ that

$$\begin{aligned} Re(f(z)/z) &\geq \frac{1}{\beta}\int_0^1 t^{\frac{1}{\beta}-1}[(1-\alpha)\frac{1-tr}{1+tr} + \alpha]dt \\ &= \frac{1}{\beta}\int_0^1 t^{\frac{1}{\beta}-1}\frac{1-(1-2\alpha)tr}{1+tr}dt. \end{aligned}$$

Noting that $|f(z)/z| \geq Re(f(z)/z)$, we can obtain the left-side inequality of (9) at once. That the function defined by (4) can attain the equality is obvious.

If we start from (6), we can obtain (10) similarly. The proof of Theorem 5 is completed.

3. A subclass of $B(\beta, \alpha)$

Owa [5] defined the subclasses $V(\theta_n)$ and $V(\theta_n, \gamma)$ of analytic functions, that is, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A$ and satisfies $arg(a_n) = \theta_n (n \geq 2)$, we call $f(z) \in V(\theta_n)$; if further there exists a constant γ such that

$$\theta_n + (n-1)\gamma \equiv \pi \pmod{2\pi}, \tag{11}$$

then we call $f(z) \in V(\theta_n, \gamma)$. Let V denote the union of $V(\theta_n, \gamma)$ obtained by taking all admissible sequences $\{\theta_n\}$ and all admissible real numbers γ.

Let $B_\alpha(\beta) = B(\beta, \alpha) \cap V$, it is a subclass of $B(\beta, \alpha)$. It is clear that $B_\alpha(0) = B_\alpha$ and $B_\alpha(1) = b_\alpha$ which were introduced by Owa [5]. In this section we study some coefficient inequalities, distortion theorems of the class $B_\alpha(\beta)$, etc.

Theorem 6. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B_\alpha(\beta)$, then*

$$\sum_{n=2}^{\infty} (1 + (n-1)\beta)|a_n| \leq 1 - \alpha, \tag{12}$$

this inequality is sharp.

Proof. When $f(z) \in B_\alpha(\beta)$, $f(z) \in B(\beta, \alpha)$, so we have inequality (1). Putting the power series expansion of $f(z)$ into (1), we have

$$Re\{1 + \sum_{n=2}^{\infty} (1 + (n-1)\beta) a_n z^{n-1}\} > \alpha.$$

Also, for $f(z) \in B_\alpha(\beta)$, we have $f(z) \in V(\theta_n, \gamma)$, let $z = re^{i\gamma}$ and let $r \to 1^-$, we obtain

$$Re\{1 + \sum_{n=2}^{\infty} (1 + (n-1)\beta)|a_n| e^{i(\theta_n + (n-1)\gamma}\} \geq \alpha.$$

Using (11) we have

$$1 - \sum_{n=2}^{\infty} (1 + (n-1)\beta)|a_n| \geq \alpha,$$

that is the inequality we need to prove. The sharpness of (12) can be seen from the following function

$$f(z) = z + \frac{1-\alpha}{1+(n-1)\beta} e^{i\theta_n} z^n \quad (n \geq 2).$$

Remark. If we let $\beta = 0$ and $\beta = 1$ in Theorem 6, we can obtain Theorem 1 and Theorem 2 of paper [5] respectively.

Theorem 7. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B_\alpha(\beta)$, then for $|z| < 1$ we have*

$$|z| - \frac{1-\alpha}{1+\beta}|z|^2 \leq |f(z)| \leq |z| + \frac{1-\alpha}{1+\beta}|z|^2 \tag{13}$$

and

$$|z| - (1-\alpha)|z|^2 \leq |(1-\beta)f(z) + \beta z f'(z)| \leq |z| + (1-\alpha)|z|^2. \tag{14}$$

The function $f(z) = z + \frac{1-\alpha}{1+\beta} e^{i\theta_2} z^2$ attains the equalities of these inequalities at $z = \pm |z| e^{-i\theta_2}$ respectively.

Proof. From Theorem 6 we have

$$(1+\beta) \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} (1+(n-1)\beta)|a_n| \leq 1-\alpha,$$

so $\sum_{n=2}^{\infty} |a_n| \leq (1-\alpha)/(1+\beta)$, hence we have

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} |a_n||z|^n \geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \geq |z| - \frac{1-\alpha}{1+\beta}|z|^2,$$

and

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} |a_n||z|^n \leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \leq |z| + \frac{1-\alpha}{1+\beta}|z|^2.$$

From these two inequalities we can obtain the inequality (13).

For the inequality (14), because

$$(1-\beta)f(z) + \beta z f'(z) = z + \sum_{n=2}^{\infty} (1+(n-1)\beta) a_n z^n,$$

from Theorem 6 we can obtain

$$\begin{aligned} |(1-\beta)f(z)+\beta z f'(z)| &\leq |z|+\sum_{n=2}^{\infty}[1+(n-1)\beta]|a_n||z|^n \\ &\leq |z|+(1-\alpha)|z|^2, \end{aligned}$$

and

$$\begin{aligned} |(1-\beta)f(z)+\beta z f'(z)| &\geq |z|-\sum_{n=2}^{\infty}[1+(n-1)\beta]|a_n||z|^n \\ &\geq |z|-(1-\alpha)|z|^2. \end{aligned}$$

These are the inequality (14). The sharpness of (13) and (14) is obvious.

Remark. If we let $\beta = 0$ and $\beta = 1$ in Theorem 7, we can obtain Theorem 3 and Theorem 4 of paper [5] respectively.

From (13) we can obtain the covering theorem of class $B_\alpha(\beta)$.

Corollary. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B_\alpha(\beta)$, then the unit disk $E = \{z : |z| < 1\}$ is mapped by $f(z)$ onto a region which contains the disk $|w| < \frac{\alpha+\beta}{1+\beta}$. This result is sharp, the extremal function is given in Theorem 7.*

Theorem 8. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B_\alpha(\beta)$, then we have $f(z) \in S^*(\frac{2\alpha+\beta-1}{\alpha+\beta})$ $(|z| < 1)$ if $\beta \neq 0$ and $2\alpha + \beta \geq 1$, that is, $f(z)$ is starlike of order $\frac{2\alpha+\beta-1}{\alpha+\beta}$ in E, thus $f(z)$ is univalent in E.*

Proof. It is sufficient for us to prove that

$$|\frac{zf'(z)}{f(z)} - 1| < 1 - \frac{2\alpha+\beta-1}{\alpha+\beta}. \tag{15}$$

It is easy to know

$$|\frac{zf'(z)}{f(z)} - 1| = |\frac{1+\sum_{n=2}^{\infty} n a_n z^{n-1}}{1+\sum_{n=2}^{\infty} a_n z^{n-1}} - 1| \leq \frac{\sum_{n=2}^{\infty}(n-1)|a_n|}{1-\sum_{n=2}^{\infty}|a_n|} (|z| = r < 1),$$

so for $\beta \neq 0$ we have

$$\beta|\frac{zf'(z)}{f(z)} - 1| - 1 \leq \frac{\sum_{n=2}^{\infty}[1+(n-1)\beta]|a_n| - 1}{1-\sum_{n=2}^{\infty}|a_n|},$$

and from Theorem 6 we obtain

$$\beta|\frac{zf'(z)}{f(z)} - 1| - 1 \leq \frac{1-\alpha-1}{1-\frac{1-\alpha}{1+\beta}} = \frac{-\alpha(1+\beta)}{\alpha+\beta},$$

that is

$$|\frac{zf'(z)}{f(z)} - 1| < \frac{1-\alpha}{\alpha+\beta} = 1 - \frac{2\alpha+\beta-1}{\alpha+\beta}.$$

The proof of Theorem 8 is completed.

Definition. The fractional integral of order λ of $f(z)$ is defined by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta, \tag{16}$$

where $\lambda > 0$, $f(z)$ is analytic in a simply connected region containing the origin in the z-plane, the power in (16) is meant as principal value.

Theorem 9. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in B_\alpha(\beta)$, then for $\lambda > 0$ and $z \in E$ we have*

$$|D_z^{-\lambda} f(z)| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \{1 - \frac{2(1-\alpha)}{(1+\beta)(2+\lambda)} |z|\} \tag{17}$$

and

$$|D_z^{-\lambda} f(z)| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \{1 + \frac{2(1-\alpha)}{(1+\beta)(2+\lambda)} |z|\}. \tag{18}$$

Inequalities (17) and (18) are sharp.

Proof. We consider the function

$$F(z) = \Gamma(2+\lambda) z^{-\lambda} D_z^{-\lambda} f(z),$$

by the aid of Gauss geometric function. Then we can obtain the following power series expansion of $F(z)$:

$$\begin{aligned} F(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} a_n z^n \\ &= z + \sum_{n=2}^{\infty} A_n z^n, \end{aligned}$$

where $A_n = (\Gamma(n+1)\Gamma(2+\lambda)/\Gamma(n+1+\lambda))a_n$. For $\lambda > 0$, $n \geq 2$,

$$0 < \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} < \frac{2}{2+\lambda}$$

and $f(z) \in B_\alpha(\beta)$, so

$$\sum_{n=2}^{\infty} |A_n| \leq \frac{2}{2+\lambda} \sum_{n=2}^{\infty} |a_n| \leq \frac{2(1-\alpha)}{(1+\beta)(2+\lambda)}.$$

Thus

$$|F(z)| \geq |z| - \sum_{n=2}^{\infty} |A_n||z|^n \geq |z| - |z|^2 \sum_{n=2}^{\infty} |A_n| \geq |z| - \frac{2(1-\alpha)}{(1+\beta)(2+\lambda)}|z|^2.$$

We can obtain (17) from this inequality, and

$$|F(z)| \leq |z| + \sum_{n=2}^{\infty} |A_n||z|^n \leq |z| + |z|^2 \sum_{n=2}^{\infty} |A_n| \leq |z| + \frac{2(1-\alpha)}{(1+\beta)(2+\lambda)}|z|^2,$$

so we can obtain (18). The sharpness of the results can be seen from the function $f(z)$ defined by

$$D_z^{-\lambda} f(z) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)}\{1 + \frac{2(1-\alpha)}{(1+\beta)(2+\lambda)} e^{i\theta_2} z\},$$

that is, by $f(z) = z + \frac{1-\alpha}{1+\beta} e^{i\theta_2} z^2$.

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