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ON A NON-SELF ADJOINT SINGLUAR BOUNDARY VALUE PROBLEM

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This paper is devoted to study a boundary value problem of Sturm Liouville type with a discontinuous density function and including a spectral parameter in the boundary condition. The kernel resolvent is constructed and the distribution of the spectrum of the considered problem is investigated. Moreover, a new approach is induced to construct the adjoint problem associated to the problem. Finally, the continuous spectrum of the problem is investigated and whence the spectrum of the adjoint problem is given.

Introduction

In the space $L_2(0,\infty,\rho(x))$ the boundary value problem

$$-y''+q(x)y=\lambda\rho(x)y, \tag{1}$$

$$y'(0)-\lambda\sum_{n=1}^{m}\alpha_n y(a_n)=0 \tag{2}$$

is considered, where the function $q(x)$ is a complex valued, continuous on $[0,\infty)$ and satisfies the condition

$$\int_0^\infty x|q(x)|dx<\infty. \tag{3}$$

The function $\rho(x)$ is discontinuous at $x=b$ and has the form

$$\rho(x)=\begin{cases}\gamma^2, & 0\le x\le b\\ 1, & b<x<\infty, \gamma\neq 1, \gamma>0.\end{cases}$$

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Also, λ is a complex parameter and α_n, a_n are real constants.

It is well known [See(5.p.146-152)] that the boundary value problems with spectral parameter in the boundary condition have many interesting applications in mathematical physics.

It should be mentioned that the spectrum of the boundary value problem (1)-(2) has been previously investigated [1] when $\rho(x) \geq \alpha > 0$ and the boundary condition $y(0) = 0$ holds.

Also, a regular boundary value problem with spectral parameter in the boundary condition was discussed in [7]. Moreover, in [6] the case of two-point boundary value problems with eigenvalue parameter in the boundary condition was studied.

In the following treatments, some certain solutions of the equation (1) under the hypothesis that condition (3) holds were shown. A detailed study of the discrete spectrum of the boundary value problem (1)-(2) was conducted where the resolvent of this boundary problem was obtained. Also, the adjoint problem of the same boundary problem was constructed. Futhermore, the continuous spectrum of the considered problem (1)-(2) and the spectrum of the adjoint problem associated with the boundary value problem (1)-(2) were investigated.

Q1. Some solutions of the equation (1)

From condition (3) it is evident that (1) reduces asymptotically to the simpler equation $-y'' = \lambda \rho y$ as $x \to \infty$. This permits us a complete investigation of the properties of the solution to equation (1).

Let $\psi(x,k)$ and $\varphi(x,k)$ denote the solutions to the equation (1) on the interval $[0,b]$ which satisfy the initial conditions

$$\psi(0,k) = 1, \quad \psi'(0,k) = 0$$
$$\varphi(0,k) = 0, \quad \varphi'(0,k) = 1,$$

where $\lambda^{\frac{1}{2}} = k = \xi + i\tau$ such that $0 \leq argk < \pi$.

Lemma 1. *The solution $\psi(x,k)$ of the equation (1) on the interval $[0,b]$ may be expressed in the form*

$$\psi(x,k) = \cos(k\gamma x) + \int_0^x A(x,t)\cos(k\gamma t)dt, 0 \leq t \leq x \leq b,$$

where the kernel $A(x,t)$ has summable derivatives A'_x, A'_t and satisfies the conditions

$$A(x,x) = \frac{1}{2}\int_0^x q(t)dt; \frac{\partial}{\partial t}A(x,t)|_{t=0} = 0.$$

Moreover

$$\psi(x,k) = \cos k\gamma x(1 + O(1/k)) \text{ as } \tau \geq 0 \text{ and } |k| \to \infty.$$

See [2].

Lemma 2. *The solution $\varphi(x,k)$ can be written on the form*

$$\varphi(x,k) = \frac{\sin k\gamma x}{k\gamma} + \int_0^x B(x,t)\frac{\sin k\gamma t}{k\gamma}dt, 0 \leq t \leq x \leq b,$$

where, the kernel $B(x,t)$ has summable derivatives B'_x, B'_t and satisfies the conditions

$$B(x,t) = \frac{1}{2}\int_0^x q(t)dt; B(x,0) = 0.$$

In addition

$$\varphi(x,k) = \frac{\sin k\gamma x}{k\gamma}(1 + O(1/k)) \text{ as } \tau \geq 0 \text{ and } |k| \to \infty.$$

Lemma 3. *If the condition (3) is satisfied, then as $x > b$ and $\tau \geq 0$ the equation (1) has the solution $f(x,k)$ which may be expressed in the form*

$$f(x,k) = \exp(ikx) + \int_x^\infty K(x,t)\exp(ikt)dt, b < x \leq t < \infty,$$

where the kernel $K(x,t)$ has continuous derivatives with respect to x and t and satisfies the inequalities

$$|K(x,t)| \leq \frac{1}{2}\exp\{\sigma_1(x)\}\sigma\frac{(x+t)}{2};$$

$$|K'_x(x,t)|, |K'_t(x,t)| \leq \frac{1}{2}|q(\frac{x+t}{2})| + \frac{1}{2}\exp\{\sigma(x)\}\sigma_1\frac{(x+t)}{2},$$

where $\sigma(x)$ and $\sigma_1(x)$ are defined by the following formulas

$$\sigma(x) = \int_x^\infty |q(t)|dt \text{ and } \sigma_1(x) = \int_x^\infty t|q(t)|dt.$$

The solution $f(x,k)$ is an analytic function of k in the upper half plane $\tau > 0$ and is continuous on the real line. This solution has the following asymptotic behaviour

$$f(x,k) = \exp(ikx)[1 + O(1/k)]$$

and $f(x,k) = ik\exp(ikx)[1+O(1/k)]$ *as* $|k| \to \infty$ *for all* x *and* $\tau \geq 0$.

This lemma and the following lemma can be proved by making use of [2].

Lemma 4. *Equation (1) has the solution* $f_1(x,k)$ *in the domain* $x > b$, $\tau \geq 0$, $|k| \geq \beta$ *and as* $|k| \to \infty$

$$f_1(x,k) = \exp(ikx)[1+O(1/k)], f_1(x,k) = -ik\exp(-ikx)[1+O(1/k)]$$

uniformly with respect to $x > b$.
This solution is a holomorphic function of k *in the domain* $\tau \geq 0$, $|k| \geq \beta$.
It should be mentioned that [5]

$$f(x,k) = \begin{cases} c_1(k)\psi(x,k) + c_2(k)\varphi(x,k), & 0 \leq x \leq b \\ \exp(ikx) + \int_x^\infty K(x,t)\exp(ikt)dt, & b < x < \infty, \end{cases} \tag{4}$$

where

$$c_1(k) = f(b,k)\varphi'(b,k) - f'(b,k)\varphi(b,k)$$

and

$$c_2(k) = f'(b,k)\psi(b,k) - f(b,k)\psi'(b,k), \tau \geq 0.$$

In addition, as $\tau \geq 0$, $|k| \to \infty$ *we have*

$$f(x,k) = \begin{cases} \exp(ikb)[cos k\gamma(x-b) + \frac{i}{\gamma}\sin k\gamma(x-b)](1+O(\frac{1}{k})), & 0 \leq x \leq b \\ \exp(ikx)[1+O(\frac{1}{k})], & b < x < \infty. \end{cases} \tag{5}$$

Q2. On the discrete spectrum and the resolvent of the boundary value problem (1)-(2)

In this section we study the discrete spectrum and construct the resolvent of the problem (1)-(2).

Now, we use the methods of the works [2,3] to prove the following theorems.

Theorem 1. *The boundary value problem (1)-(2) has no positive eigenvalues.*

Theorem 2. *The necessary and sufficient conditions that* $\lambda \neq 0$ *be an eigenvalue of the boundary value problem (1)-(2) are*

$$\lambda = k^2; \tau > 0; W(k) = f'(0,k) - k^2\sum_{n=1}^{m}\alpha_n f(a_n,k) = 0 \tag{6}$$

Here, the asymptotics behaviour for the eigenvalues of the problem (1)-(2) is then investigated.

In view of (4), (5) and (6) it can be found that

$$\begin{aligned} W(k) &= k\gamma[\sin kb\gamma + \frac{i}{\gamma}\cos k\gamma b]\exp(ikb)[1+O(\frac{1}{k})] \\ &\quad -k^2\sum_{n=1}^{r}\alpha_n \exp(ikb)[\cos k\gamma(a_n-b)+\frac{i}{\gamma}\sin k\gamma(a_n-b)] \\ &\quad \cdot[1+0(\frac{1}{k})] - k^2\sum_{n=r+1}^{m}\alpha_n\exp(ika_n)[1+O(\frac{1}{k})]. \end{aligned}$$

The question concerning the discrete spectrum of the boundary value problem (1)-(2) leads to the study of zero's of the function $W(k)$ in the upper half plane $\{k : \tau > 0\}$.

As long as the function $W(k)$ is holomorphic in the upper half plane, then the set of zero's of the function $W(k)$ has no more than countable set $\{k_n\}$. The latter set has thus been classified:

The limit points of the set exists on the real axis and at infinity. If the condition $\exp(\varepsilon x)q(x) \in L_1(0,\infty)$ holds then $W(k)$ is holomorphic in the half plane $M_\varepsilon = \{k : \tau > -\frac{\varepsilon}{2}\}$ and therefore $\{k_n\}$ is impossible to have limit points on the real axis. Accordingly the study of the discrete spectrum classification is equivalent to the study of the zero's function

$$\begin{aligned} W_0(k) &= k\gamma[\sin k\gamma b + \frac{i}{\gamma}\cos k\gamma b]\exp(ikb) \\ &\quad -k^2\sum_{n=1}^{r}\alpha_n[\cos k\gamma(a_n-b)+\frac{i}{\gamma}\sin k\gamma(a_n-b)]\exp(ikb) \\ &\quad -k^2\sum_{n=r+1}^{m}\alpha_n\exp(ika_n) \end{aligned}$$

from the upper half plane. Here, some different examples are given.

Example 1. If $\rho(x)=1$ and the boundary condition has the form $y'(0)-k^2y(1)=0(2')$, then

$$W(k) = ik - k^2\exp(ik) = 0.$$

Therefore, it is found that $k_n = 2\pi n + i\ln|2\pi n| + \alpha_n + 0(1)$. This means that the problem (1)-(2') has an infinite number of eigenvalues.

Example 2. If $\rho(x) \equiv 1$ and the boundary condition has the form

$$y'(0) - k^2[y(1) + y(2)] = 0 \tag{2''}$$

thus $W(k) = ik - k^2[\exp(ik) + \exp(2ik)] = 0$ and

$$W_0(k) = \exp(ik) + \exp(2ik) = 0.$$

Theorefore

$$k_n^0 = 2n\pi + \pi.$$

Hence

$$k_n = k_n^0 + \varepsilon_n.$$

The quantity ε_n can be calculated more accurately. Hence

$$k_n = 2n\pi + \pi + \frac{1}{\pi + 2n\pi} + \frac{3}{2i}(\frac{1}{\pi + 2n\pi})^2 + O(\frac{1}{n^3});$$

and since $Imk_n = -\frac{3}{2}(\frac{1}{\pi+2n\pi})^2 + O(\frac{1}{n^3})$ and $\tau_n < 0$ for large n, it follows that the problem (1)-(2") could have a finite number of eigenvalues k approaching the real axis as $n \to \infty$. Hence, if $q(x) \neq 0$ then it is possible to take these numbers as eigenvalues. These examples show the difficulty of the study of the distribution of eigenvalues.

In the forthcoming work this question can be studied in detailed.

Theorem 3. *All numbers $\lambda = k^2, \tau > 0$ and $W(k) \neq 0$ belong to the resolvent set of the problem (1)-(2). If $W(k) \neq 0$ then the resolvent R_λ is an integral operator*

$$R_\lambda(\rho f) = \int_0^\infty R(x, t, k)\rho(t)f(t)dt,$$

with the kernel

$$R(x, t, k) = R_0(x, t, k) + \frac{k^2 f(x, k)}{W(k)} \sum_{n=1}^{m} \alpha_n R_0(a_n, t, k), \tag{7}$$

where

$$R_0(x, t, k) = -\frac{1}{f'(0, k)} \begin{cases} f(x, k)\psi(t, k), & t \leq x \\ \psi(x, k)f(t, k), & t \geq x. \end{cases}$$

Proof. It follows immediately by theorem 2 that all numbers $\lambda = k^2, W(k) \neq 0$, $\tau > 0$ belong to the resolvent set of the problem (1)-(2). Now, by assumption, $\lambda = k^2$ is not an eigenvalue of the problem (1)-(2), thus the resolvent R_λ exists. This means that there exists a solution of the equation

$$-y'' + q(x)y - \lambda\rho(x)y = \rho f \tag{8}$$

belongs to $L_2(0,\infty,\rho(x))$ and satisfying the condition (2). Denote by $y_0(x,k)$ to the solution of equation (8) belongs to $L_2(0,\infty,\rho(x))$ and satisfies the condition $y'(0)=0$. This solution has the form

$$y_0(x,k) = \int_0^\infty R_0(x,t,k)\rho(t)f(t)dt,$$

where

$$R_0(x,t,k) = -\frac{1}{f'(0,k)}\begin{cases} f(x,k)\psi(t,k), & t \le x \\ \psi(x,k)f(t,k), & t \ge x. \end{cases}$$

Hence, the general solution of equation (8) from $L_2(0,\infty,\rho(x))$ can be written on the form [2]

$$y(x,k) = y_0(x,k) + Cf(x,k), \tag{9}$$

where C is an arbitrary constant. Since, the function $y(x,k)$ satisfies the condition (2) thus we have

$$C = \frac{k^2\sum_{n=1}^{m}\alpha_n y_0(a_n,k)}{W(k)}.$$

Substituting in (9) to get

$$y(x,t) = \int_0^\infty R(x,t,k)\rho(t)f(t)dt,$$

where $R(x,t,k)$ is defined by the formula (7).

Q3. Construction of the adjoint problem of the problem (1)-(2)

In this section the adjoint problem of the problem (1)-(2) is constructed.

Now, suppose that $z(x) \in L_2(0,\infty,\rho(x))$ and $z'(x)$ exist and absolutely continuous in the entire intervals $[0,a_1),(a_1,a_2),\cdots,(a_{n-1},a_n),(a_n,\infty)$.

Theorem 4. *Let the previous conditions be satisfied. Then the adjoint problem of the problem (1)-(2) has the form*

$$-z'' + \overline{q(x)}z = \lambda\rho z, \tag{10}$$

where

(i) *the function $z(x)$ is absolutely continuous on $(0,\infty)$;*

(ii) *the function $z(x)$ has a continuous derivative in all entire intervals $[0,a_1),(a_1,a_2),\cdots,(a_n,\infty)$;*

(iii) $z'(0)=0$;

(iv) $\overline{z'}(a_n+0)-\overline{z'}(a_n-0)=\lambda\alpha_n\overline{z}(0)$;

(v) *when $x\neq a_n, n=\overline{1,m}$ the function $z(x)$ has a second derivative and satisfies the equation*

$$-\overline{z''}+q(x)\overline{z}=\overline{\lambda}\rho\overline{z}+f.$$

It should be mentioned that the concept of satisfying equation (10) can be understood in the sense of generalized functions.

Proof. We construct the adjoint problem by using the kernel resolvent (7). It is well known from the theory of operators [2,4] that the domain of definition of the adjoint problem can be defined by the resolvent. Here, we introduce the set $D(L^*_{\overline{\lambda}})$ such that

$$z(x)=\int_0^\infty \overline{R(t,x,\sqrt{\overline{\lambda}})\rho(t)f(t)dt},$$

where f is an arbitrary function from $L_2(0,\infty\rho(x))$. By virtue of the evident equality

$$[R_{\overline{\lambda}}(A)]^*=[(A-\overline{\lambda}\rho)^{-1}]^*=(A^*-\lambda\rho)^{-1}=R_\lambda(A^*),$$

it yields that $D(L^*_{\overline{\lambda}})$ coincide with domain of definition of the adjoint problem. We study the properties of the functions belonging to the set $D(L^*_{\overline{\lambda}})$.

i) The function $z(x)\in D(L^*_{\overline{\lambda}})$ is an absolutely continuous on $[0,\infty)$. In fact

$$\begin{aligned}\overline{z}(x) &= \int_0^\infty R_0(t,x,\sqrt{\overline{\lambda}})\rho(t)f(t)dt\\ &+\frac{\overline{\lambda}}{W(\sqrt{\overline{\lambda}})}\sum_{n=1}^{m}\alpha_nR_0(a_n,x,\sqrt{\overline{\lambda}})\int_0^\infty f(t,\sqrt{\overline{\lambda}})\rho(t)f(t)dt \quad (11)\end{aligned}$$

Here, the first quantity is an absolutely continuous function and the second quantity also satisfies this fact in view of the properties of the kernel $R_0(a_n,x,\sqrt{\overline{\lambda}})$.

ii) The function $z'(x)$ has a continuous derivative on an any of intervals $[0,a_1),(a_1,a_2),\cdots,(a_n,\infty)$. The first quantity in (11) has a continuous derivative on $[0,\infty)$ and in the second quantity every function from $R_0(a_n,x,\sqrt{\bar\lambda})$ has a continuous derivative on $[0,a_n)$ and $(a_n,\infty), n=\overline{1,m}$. Moreover

$$\frac{\partial}{\partial x}R_0(a_n,x,\sqrt{\bar\lambda})|_{x=a_n+c}-\frac{\partial}{\partial x}R_0(a_n,x,\sqrt{\bar\lambda})|_{x=a_n-0}=-1.$$

From this equality the required statement is satisfied

iii) The condition $z'(0)=0$ holds.

This is evident from the equality

$$\frac{\partial}{\partial x}R_0(a_n,x,\sqrt{\bar\lambda})|_{x=0}=0$$

iv) The following formula holds

$$\begin{aligned}\overline{z'}(a_n+0)-\overline{z'}(a_n-0) &= -\frac{\alpha_n\bar\lambda}{W(\sqrt{\bar\lambda})}\int_0^\infty f(t,\sqrt{\bar\lambda})\rho f(t)dt\\ &= \bar\lambda\alpha_n\overline{z}(0).\end{aligned}$$

In fact,

$$\begin{aligned}z(0) &= \int_0^\infty R_0(t,0,\sqrt{\bar\lambda})\rho(t)f(t)dt\\ &\quad+\frac{\bar\lambda}{W(\sqrt{\bar\lambda})}\sum_{n=1}^m\alpha_nR_0(a_n,0,\sqrt{\bar\lambda})\int_0^\infty f(t,\sqrt{\bar\lambda})\rho(t)f(t)dt\\ &= -\frac{1}{f'(0,\sqrt{\bar\lambda})}[\int_0^\infty f(t,\sqrt{\bar\lambda})\rho(t)f(t)dt\\ &\quad+\frac{\bar\lambda}{W(\sqrt{\bar\lambda})}\sum_{n=1}^m\alpha_nf(a_n,\sqrt{\bar\lambda})\int_0^\infty f(t,\sqrt{\bar\lambda})\rho(t)f(t)dt]\\ &= -\int_0^\infty f(t,\sqrt{\bar\lambda})\rho(t)f(t)dt[W(\sqrt{\bar\lambda})+\bar\lambda\sum_{n=1}^m\alpha_nf(a_n,\sqrt{\bar\lambda})]\\ &= -\frac{1}{W(\sqrt{\bar\lambda})}\int_0^\infty f(t,\sqrt{\bar\lambda})\rho(t)f(t)dt.\end{aligned}$$

Then, the last equality is equivalent to the required equality.

v) Let $x\neq a_n, n=\overline{1,m}$ and the function $z(x)$ has a second derivative and satisfies the equation

$$-\bar z''+q(x)\bar z=\bar\lambda\rho\bar z+f.$$

In fact, the first quantity in the expression of $z(x)$ satisfies this equation and the second quantity by using the defintion of kernel resolvent satisfies the associated homogenous equation for all $x \neq a_n$, $n = \overline{l, m}$.

Note. By the concepts of the generalized functions the adjoint problem of the problem (1)-(2) can be written in the following form

$$-z'' + \overline{q(x)}z + \lambda \sum_{n=1}^{m} \alpha_n \delta(x - a_n) z(0) = \lambda \rho z \tag{12}$$

$$z'(0) = 0, \tag{13}$$

where $z(x)$ satisfies the properties (i)-(v) from the theorem 6.

Q4. The continuous spectrum of (1)-(2) and the spectrum of the adjoint problem (12)-(13)

In this section we investigate the continuous spectrum of (1)-(2) and whence we obtain the sepctrum of the adjoint problem (12)-(13).

Theorem 5. *The continuous spectrum of the problem (1)-(2) lies on the semi axis* $\lambda \geq 0$.

Proof. If $\varepsilon < \frac{1}{2}$, it found that $A > b$ and in view of (4) it is possible to get $f(x,k) \geq \exp ikx(1-\varepsilon)$ for $x \geq A > 1$. Suppose that $f(x) \equiv 0$ as $x \notin [A, A+1]$ and $f(x) \in L_2(0, \infty)$. then

$$R(\rho f) = \int_A^{A+1} R_0(x,t,k) f(t) dt + \frac{k^2}{W(k)} \sum_{n=1}^{m} \alpha_n f(x,k) \int_A^{A+1} R_0(a_n,t,k) f(t) dt.$$

If it is assumed that the number $A \geq a_n$, $n = \overline{l, m}, x > A+1$, thus

$$R(\rho f) = \int_A^{A+1} [\frac{-\psi(t,k)}{f'(0,k)} + \frac{k^2}{W(k)} f(x,k) \sum_{n=1}^{m} R_0(a_n,t,k)] f(t) dt.$$

Now let

$$\bar{f}(t) = \begin{cases} -\frac{\psi(t,k)}{f'(0,k)} + \frac{k^2}{W(k)} \sum_{n=1}^{m} R_0(a_n,t,k)], & t \in [A, A+1] \\ 0, & t \notin [A, A+1] \end{cases}$$

Then

$$\int_A^B |R(f)|^2 dx = \int_A^B |f(x,k)|^2 dx \cdot \int_A^{A+1} |f(t)|^2 dt.$$

As $B \to \infty$ it yields that

$$\begin{aligned}\int_A^\infty |R(f)|^2 dx &\geq (1-\varepsilon)^2 \int_A^\infty |\exp(ikx)|^2 dx \int_A^{A+1} |f(t)|^2 dt \\ &= \frac{(1-\varepsilon)^2}{2\tau} \exp(-2\tau A) \int_A^{A+1} |f(t)|^2 dt.\end{aligned}$$

Therefore, when $\tau \to 0$ the norm of the resolvent is increased. Hence, the half line $[0, \infty)$ coincides with the continuous spectrum of operator L_λ.

Now, from [3,4] the following results are attained.

Theorem 6. *The spectrum of the adjoint problem (12)-(13) consists of the eigenvalues* $\bar{\lambda}$ *when* $W(\lambda^{\frac{1}{2}}) = 0$, $Im\lambda > 0$ *and the continuous spectrum lies on the semi axis* $[0, \infty)$.

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