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Bayesian Prediction Inferences for the Burr Model Under the Random Censoring

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ABSTRACT

Using a noninformative prior and a gamma prior, the Bayesian predictive density and the prediction intervals for a future observation or the p–th order statistic of n' future observations from the Burr distribution have been obtained. In additions, we examine the sensitivities of the results to the choice of model.

1. Introduction

Statistical prediction analysis may provide warranty limits for the future performance of systems or may be used in situations where a producer compare the performance of both his product and that of a competitor and wishes to determine the difference in future mean performance of the products. Thus statistical prediction plays a very important role in the reliability analysis for some lifetime models, quality control and many other application areas.

The problem of predicting a future observation has received much attention and has been dealt mainly in two approaches. One is the usual classical approach and the other is Bayesian approach. The classical approach for this problem uses a

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pivotal quantity method, a maximum likelihood estimation method and a method based on the idea of a conditioning on a sufficient statistic method *etc.*. Patel(1989) reviewed some known results on prediction intervals for univariate distributions.

Based on the Bayesian approach, Dunsmore(1974) discussed this problem when the underlying distribution is one or two parameter exponential distribution. Lingappaiah(1983) developed the Bayesian approach to the prediction problem in the exponential population. Chhikara and Guttman(1982), Sinha(1989), Nigm and Hamdy(1987) suggested the Bayesian inference about prediction for inverse Gaussian, lognormal and Pareto distribution, respectively.

Clarotti and Spizzichino(1989) proposed the Bayesian predictive appoach in reliability theory, and Csenki(1990) discussed the Bayesian predictive analysis of fundamental software reliability model.

Let $\underline{X} = (X_1, X_2, \cdots, X_n)$ be an observed random sample of size n from the distribution function $F(\cdot|\mu)$ with probability density function $f(\cdot|\mu)$ where μ may be a real or vector valued parameter. Let $\underline{Y} = (Y_1, Y_2, \cdots, Y_n)$ be the second independent random sample of future observations of size n' from the same distribution. Let $\Pi(\cdot)$ be a prior distribution of μ and also let $\Pi(\mu|\underline{x})$ be the posterior distribution of μ given $\underline{X} = \underline{x}$. Then the predictive density function for a future observation y is given by

$$\Pi(y|\underline{x}) = K \int_{\mathcal{M}} f(y|\mu)\Pi(\mu|\underline{x})\, d\mu,$$

where $\mathcal{M}$ is the range space of μ and K is the normalizing constant. So the idea of Bayesian predictive inference is the average of the likelihood of a future observation based on one updated posterior density of μ given $\underline{X} = \underline{x}$.

The $100(1-\gamma)\%$ equal-tail prediction interval (C_1, C_2) for future observation is

$$\int_{-\infty}^{C_1} \Pi(y|\underline{x})\, dy = \int_{C_2}^{\infty} \Pi(y|\underline{x})\, dy = \frac{\gamma}{2}.$$

The prediction interval C is said to be a most plausible Bayesian prediction interval of cover κ if C has the form

$$C = \{y : \Pi(y|\underline{x}) \geq \gamma\},$$

where γ is determined by

$$\Pi(C|\underline{x}) = \kappa.$$

In Section 2, we derive the predictive density and the prediction intervals of a future observation and the p–th order ststistic of n' future observations for the Burr distribution under random censoring. As a prior distribution, we consider a noninformative prior and a gamma prior.

In Section 3, using some results obtained in this paper, we consider the study on model–sensitivity and concluding remarks are given.

2. Predictive densities and prediction intervals under random censoring

Let $\underline{X} = (X_1, X_2, \cdots, X_n)$ be a random sample from the Burr distribution with unknown parameter θ and known parameter b whose the probability density function is given by

$$f(x|\theta, b) = \theta b x^{b-1}(1+x)^{-(1+\theta)}, x > 0. \tag{2.1}$$

Under the random censoring scheme, the likelihood function is

$$\begin{aligned} L(\theta|\underline{x}, b) &= \prod_{i \in D_1} f(x_i) \prod_{i \in D_2} \overline{F}(x_i) \\ &= (\theta b)^{|D_1|} \prod_{i \in D_1} [(x_i^{b-1}(1+x_i^b)^{-(\theta+1)}] \prod_{i \in D_2} (1+x_i^b)^{-\theta}. \end{aligned} \tag{2.2}$$

where D_1 is the set of individuals for whom lifetimes are obtained and D_2 is the set of individuals for whom only censoring times are avalible.

Now, a noninformative prior for θ is assumed which is given by

$$\Pi(\theta) \propto \frac{1}{\theta}, \qquad \theta > 0. \tag{2.3}$$

Then the posterior probability density function of θ given $\underline{X} = \underline{x}$ is

$$\Pi(\theta|\underline{x}) = \frac{\{\sum_{i\in D_1}\log(1+x_i^b)+\sum_{i\in D_2}\log(1+x_i^b)\}^{|D_1|}}{\Gamma(|D_1|)} \times \theta^{|D_1|-1}\exp\left\{-\theta\left(\sum_{i\in D_1}\log(1+x_i^b)+\sum_{i\in D_2}\log(1+x_i^b)\right)\right\}. \tag{2.4}$$

It is assumed that the distribution of a future observation y given θ follows as the same Burr distribution with parameters θ and b. Then one can obtain the following theorem :

Theorem 2.1. Under a noninformative prior distribution, the predictive density function of y is as follows :

$$\Pi(y|\underline{x}) = \frac{b|D_1|y^{b-1}R^{|D_1|}}{(1+y^b)\{R+\log(1+y^b)\}^{|D_1|+1}}, \quad y>0, \tag{2.5}$$

where

$$R = \sum_{i\in D_1}\log(1+x_i^b)+\sum_{i\in D_2}\log(1+x_i^b).$$

Proof. From the posterior density function of θ given $\underline{X} = \underline{x}$ and the distribution of a future observation y, the joint probability density function of θ and y is given by

$$\Pi(y,\theta) = \frac{b|D_1|y^{b-1}R^{|D_1|}}{\Gamma(|D_1|)(1+y^b)}\theta^{|D_1|}\exp\{-\theta\left(R+\log(1+y^b)\right)\}.$$

Now by transforming $W_B = R + \log(1+y^b)$, one can prove the theorem easily.

For a noninformative prior distribution, the $100(1-\gamma)\%$ prediction bound $(C_{NL}^{**}, C_{NU}^{**})$ for a future observation is given by

$$\left(\left[\exp\{R((1-\gamma/2)^{-1/|D_1|}-1)\}-1\right]^{1/b}, \left[\exp\{R((\gamma/2)^{-1/|D_1|}-1)\}-1\right]^{1/b}\right).$$

Also the $100(1-\gamma)\%$ most plausible Bayesian prediction interval $(M_{NL}^{**}, M_{NU}^{**})$ for y can be obtained by solving the following equations :

$$\left\{\frac{R}{R+\log(1+{M_{NL}^{**}}^{b})}\right\}^{|D_1|} - \left\{\frac{R}{R+\log(1+{M_{NU}^{**}}^{b})}\right\}^{|D_1|} = 1-\gamma$$

and

$$\{R+\log(1+{M_{NU}^{**}}^{b})\}^{|D_1|+1} = \{R+(1+{M_{NL}^{**}}^{b})\}^{|D_1|+1}.$$

Instead of using a noninformative prior distribution for θ, we may take a gamma distribution with parameters α and β as a prior for θ. Then the posterior distribution of θ is

$$\Pi(\theta|\underline{x}) = \frac{(R+\beta)^{|D_1|+\alpha}}{\Gamma(|D_1|+\alpha)}\theta^{|D_1|+\alpha+1}\exp\{-\theta(R+\beta)\} \tag{2.6}$$

and one can obtain the following theorem.

Theorem 2.2. For a gamma prior distribution with parameters α and β for θ, the predictive density function for a future observation y is given by

$$\Pi(y|\underline{x}) = \frac{by^{b-1}(|D_1|+\alpha)(R+\beta)^{|D_1|+\alpha}}{\{R+\log(1+y^b)+\beta\}^{|D_1|+\alpha+1}}\exp\{-\log(1+y^b)\}, \quad y>0. \tag{2.7}$$

If a gamma prior with parameters α and β for θ is used, then the $100(1-\gamma)\%$ prediction bound $(C_{GL}^{**}, C_{GU}^{**})$ of a future observation is

$$\left(\left[\exp\{(R+\beta)(1-\gamma/2)^{-1/|D_1|+\alpha}-1)\}-1\right]^{1/b},\right.$$
$$\left.\left[\exp\{(R+\beta)((\gamma/2)^{-1/|D_1|+\alpha}-1)\}-1\right]^{1/b}\right).$$

Furthermore, the most plausible Bayesian prediction limits C_{GL}^{**} and $C_{GU}^{**}$for y are the simultaneous solutions of

$$\left\{\frac{R+\beta}{R+\log(1+M_{GL}^{**\,b})+\beta}\right\}^{|D_1|+\alpha}-\left\{\frac{R+\beta}{R+\log(1+M_{GU}^{**\,b})+\beta}\right\}^{|D_1|+\alpha}=1-\gamma$$

and

$$\{R+\log(1+M_{GU}^{**\,b})+\beta\}^{|D_1|+\alpha+1}=\{R+\log(1+M_{GL}^{**\,b})+\beta\}^{|D_1|+\alpha+1}.$$

Now we consider the predictive density function of the p–th order statistics, $Y_{(p)}$, of n' future observations when the distribution follows the Burr distribution. Here the probability density function of $Y_{(p)}$ is as follows :

$$\begin{aligned} f(y_{(p)}|\theta,b) &= \frac{\theta b y_{(p)}^{b-1}}{B(n'-p+1,p)}\sum_{i=0}^{p-1}\binom{p-1}{i}(-1)^i \\ &\quad\times(1+y_{(p)}^b)^{-\{\theta(n'-p+1+i)+1\}},\quad y_{(p)}>0. \end{aligned} \tag{2.8}$$

Therefore the following theorem can be obtained.

Theorem 2.3. Under a noninformative prior for θ, the predictive density function for the p–th ordered lifetime, $Y_{(p)}$, in a future sample of n' items is given by

$$\begin{aligned} \Pi(y_{(p)}|\underline{x}) &= \frac{b|D_1|y_{(p)}^{b-1}R^{|D_1|}}{B(n'-p+1)(1+y_{(p)}^b)}\sum_{i=0}^{p-1}\binom{p-1}{i}(-1)^i \\ &\quad\times\{R+(n'-p+1+i)\log(1+y_{(p)}^b)\}^{-(|D_1|+1)},\quad y_{(p)}>0. \end{aligned} \tag{2.9}$$

With a noninformative prior of θ, the $100(1-\gamma)\%$ equal-tail prediction limits C_{NL}^{**} and C_{NU}^{**} for $Y_{(p)}$ are the solutions of the following equations :

$$\frac{\gamma}{2} = \frac{1}{B(p, n'-p+1)} \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i (n'-p+1+i)^{-1} \times \left\{ 1 - \left(\frac{R}{(n'-p+1+i)\log(1+C_{NL}^{**b}) + R} \right)^{|D_1|} \right\}$$

and

$$\frac{\gamma}{2} = \frac{1}{B(p, n'-p+1)} \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i (n'-p+1+i)^{-1} \times \left(\frac{R}{(n'-p+1+i)\log(1+C_{NU}^{**b}) + R} \right)^{|D_1|}$$

Also the $100(1-\gamma)\%$ most plausible prediction interval $(M_{NL}^{**}, M_{NU}^{**})$ of the p-th order statistic of n' future observations can be obtained by solving the following equations :

$$\frac{1}{B(p, n'-p+1)} \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i (n'-p+1+i)^{-1} \times \left(\frac{R}{R + (n'-p+1+i)\log(1+M_{NL}^{**b})} \right)^{|D_1|}$$
$$- \frac{1}{B(p, n'-p+1)} \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i (n'-p+1+i)^{-1} \times \left(\frac{R}{R + (n'-p+1+i)\log(1+M_{NU}^{**b})} \right)^{|D_1|}$$
$$= 1 - \gamma$$

and

$$\sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i (n'-p+1+i)^{-1} \times \{R + (n'-p+1+i)\log(1+M_{NL}^{**b})\}^{-(|D_1|+1)}$$

$$= \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i (n' - p + 1 + i)^{-1}$$
$$\times \{R + (n' - p + 1 + i) \log(1 + M_{NU}^{**b})\}^{-(|D_1|+1)}.$$

Theorem 2.4. If a prior of θ is a gamma distribution with parameters α and β, then the predictive density function of $Y_{(p)}$ is as follows:

$$\Pi(y_{(p)}|\underline{x}) = \frac{b(|D_1| + \alpha) y_{(p)}^{b-1} (R + \beta)^{|D_1|+\alpha}}{B(n' - p + 1, p)(1 + y_{(p)}^b)} \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i \tag{2.10}$$
$$\times \{R + (n' - p + 1 + i) \log(1 + y_{(p)}^b) + \beta\}^{-(|D_1|+\alpha+1)}, \quad y_{(p)} > 0.$$

With a gamma prior for θ, the $100(1-\gamma)\%$ equal-tail prediction limits C_{GL}^{**} and C_{GU}^{**} for the p-th order statistic of n' future observations can be obtained by solving the equations :

$$\frac{\gamma}{2} = \frac{1}{B(n' - p + 1, p)} \sum_{i=0}^{p-1} \binom{P-1}{i} (-1)^i (n' - p + 1 + i)^{-1}$$
$$\times \left\{ 1 - \left(\frac{R + \beta}{R + (n' - p + 1 + i) \log(1 + C_{GL}^{**b}) + \beta} \right)^{|D_1|+\alpha} \right\}$$

and

$$\frac{\gamma}{2} = \frac{1}{B(p, n' - p + 1)} \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i (n' - p + 1 + i)^{-1}$$
$$\times \left(\frac{R + \beta}{R + (n' - p + 1 + i) \log(1 + C_{GU}^{**b}) + \beta} \right)^{|D_1|+\alpha}$$

Also the $100(1-\gamma)\%$ most plausible prediction interval $(M_{GL}^{**}, M_{GU}^{**})$ of $Y_{(p)}$ can obtained by solving the following equations :

$$\frac{1}{B(p, n' - p + 1)} \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i (n' - p + 1 + i)^{-1}$$

$$\times \left(\frac{R+\beta}{R+(n'-p+1+i)\log(1+M_{GL}^{**b})+\beta} \right)^{|D_1|+\alpha}$$

$$- \frac{1}{B(p,n'-p+1)} \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i (n'-p+1+i)^{-1}$$

$$\times \left(\frac{R+\beta}{R+(n'-p+1+i)\log(1+M_{GU}^{**b})+\beta} \right)^{|D_1|+\alpha}$$

$$= 1-\gamma$$

and

$$\sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i (n'-p+1+i)^{-1}$$

$$\times \{R+(n'-p+1+i)\log(1+M_{GL}^{**b})+\beta\}^{-(|D_1|+\alpha+1)}$$

$$= \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i (n'-p+1+i)^{-1}$$

$$\times \{R+(n'-p+1+i)\log(1+M_{GU}^{**b})+\beta\}^{-(|D_1|+\alpha+1)}.$$

3. The Study on the Model–sensitivity

In order to study the model–sensitivity under the random censoring, the Burr distribution with unknown parameter θ and known parameter b is considered. The noninformative prior distribution as the prior for θ is given for the values of $b = 2$, 3, 4, 5. To generate the data set of X, values of $\theta = 3$, 4, 5, 6, 7 were given. Figure 3.1 shows the predictive densities with b =2, 3, 4, 5 for the cases of $\theta = 3$ and 6, respectively. For both $\theta = 3$ and $\theta = 6$, the peakedness increases and variance decreases as the value of b is increases from 2 to 5. Also the changes of the predictive densities are more rapid for the case of $\theta = 6$ than those of the case of $\theta = 3$. Thus it can be said that the predictive density is very sensitive to the choice of values of b. One can also expect that the prediction intervals may be changed as the values of b are changed. To see this for the same values of θ and b, both the 95% equal–tail and most plausible prediction intervals were computed and given in Table 3.1. From Table 3.1, one can see the same results as those of for the predictive densities.

Figure 3.1 Predictive Density Functions of y for the Burr Distribution under the Random Censoring

Table 3.1 95% **Prediction Interval of y for the Burr Distribution under the Random Censoring**

θ	b	2	3	4	5
3	M.P.	(.0202, .7241)	(.0912, .5657)	(.1445, .5170)	(.1751, .4723)
	E.T.	(.0576, .8194)	(.1056, .5846)	(.1456, .5181)	(.1703, .4681)
4	M.P.	(.0165, .4905)	(.0632, .3769)	(.0920, .3255)	(.1066, .2865)
	E.T.	(.0410, .5459)	(.0714, .3873)	(.0922, .3256)	(.1035, .2838)
5	M.P.	(.0133, .3677)	(.0467, .2753)	(.0655, .2315)	(.0750, .2017)
	E.T.	(.0314, .4065)	(.0524, .2824)	(.0656, .2315)	(.0729, .1998)
6	M.P.	(.0111, .2929)	(.0366, .2151)	(.0505, .1785)	(.0576, .1548)
	E.T.	(.0253, .3226)	(.0410, .2206)	(.0506, .1785)	(.0559, .1533)
7	M.P.	(.0094, .2428)	(.0390, .1761)	(.0411, .1449)	(.0466, .1252)
	E.T.	(.0211, .2669)	(.0336, .1805)	(.0411, .1450)	(.0452, .1240)

M.P. : Most plausible prediction bound
E.T. : Equal–tail prediction bound

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