Youngnam Statistical Letters
June 1993, Vol. 4, 87-95

Simultaneous Estimation of Several Poisson Means under a Linex Loss Function[1]

In-Suk Lee, Won-Tae Jeong and Hye-Jeong Jeong[2]

Abstract

We find a class of admissible Bayes estimator for the mean vector $\theta = (\theta_1, \theta_2, \cdots, \theta_p)$ of Poisson distribution under a LINEX loss function.

The Monte Carlo Simulation is performed to compare the emprical Bayes estimator under the LINEX loss function and weighted squared error loss respectively.

1. Introduction

Let $X_1, X_2, \cdots, X_p$ be mutually independent Poisson random variables with means $\theta_1, \theta_2, \cdots, \theta_p$, respectively, where $\theta_i \in (0, \infty)$ is unknown for each $i = 1, 2, \cdots, p$. The restriction to one observation from each distribution involves no loss of generality because if X_{ij} $(j = 1, 2, \cdots, n_i \quad : i = 1, 2, \cdots, p)$ are mutually independent where X_{ij} $(j = 1, 2, \cdots, n_i)$ are independently and identically distributed Poisson $\mathcal{P}(\theta_i)$ $(i = 1, 2, \cdots, p)$, then the minimal sufficient statistic for $\theta = (\theta_1, \theta_2, \cdots, \theta_p)$ is $X = (X_1, X_2, \cdots, X_p)$ where $X_i = \sum_{j=1}^{n_i} X_{ij}$ is $\mathcal{P}(n_i\theta_i), (i = 1, 2, \cdots, p)$.

[1] The paper was supported by Non Directed Research Fund, Korea Rearch Foundation, 1992.

[2] Department of Statistics, Kyungpook National University, Taegu 702-701, Korea.

Ghosh(1983) obtain the class of admissible Bayes estimator of θ under the loss function

$$L_s(\Theta,\delta) = \sum_{i=1}^{p} \frac{(\delta_i - \theta_i)^2}{\theta_i}, \tag{1.1}$$

where Θ is the random variable of unknown parameter vector θ. In this paper, we consider estimation of θ using a *linearly and exponential* (LINEX) loss function.

$$L_l(\Theta,\delta) = \sum_{i=1}^{p} b\left\{e^{a(\delta_i-\theta_i)} - a(\delta_i - \theta_i) - 1\right\} \tag{1.2}$$

where $a \neq 0$ is a shape parameter and $b > 0$ is a scale parameter. This useful and flexible class of loss function was introduced by Varian(1975) and was extensively discussed by Zellner(1986).

The sign of the shape parameter a reflects the direction of the asymmetric $a > 0$ $(a < 0)$ if overestimation is more (less) serious than underestimation; and the magnitude of a reflects the degree of the asymmetry.

For small value of $|a|$, the loss is almost symmetric and merely quadratic. For the case of $p = 1$, Sadooghi-Alvandi(1990) obtained the admissible estimator of θ under the loss function (1.2).

In this paper, first, we shall find a Bayes estimator and empirical Bayes estimator of θ under the loss function (1.2). Also, we shall show the admissibility of this Bayes estimator. Next we shall compare these estimators and estimators of Ghosh.

2. Bayes estimator

Let $X_1, X_2, \cdots, X_p$ be mutually independent Poisson random variables with parameters $\theta_1, \theta_2, \cdots, \theta_p$, respectively, where $\theta_i \in (0,\infty)$ for $i = 1, 2, \cdots, p$.

Now we consider the case that $\theta_1, \theta_2, \cdots, \theta_p$ are mutually independent random variables with the p.d.f $f(\theta_j|u)$, $j = 1, 2, \cdots, p$. On $u \in (0,1)$;

$$f(\theta_j|u) = \frac{exp\left\{\frac{-u}{1-u}\theta_j\right\}\theta_j^{k_j-1}(\frac{u}{1-u})^{k_j}}{\Gamma(k_j)}, \tag{2.1}$$

where $k_j > 0$. Let $x = (x_1, x_2, \cdots, x_p)$ be observed values of random vector $X = (X_1, \cdots, X_p)$. Then the joint p.d.f of X and Θ is

$$g(x, \theta|u) = \Pi_{j=1}^{p} \frac{exp\left\{\frac{-1}{1-u}\theta_j\right\}\theta_j^{x_j+k_j-1}\left(\frac{u}{1-u}\right)^{k_j}}{\Gamma(x_j+1)\Gamma(k_j)}$$

Hence the marginal distribution of $X_1, \cdots, X_p$ are independently negative binomial distribution denoted by NB$(k_j|u)$ for $j = 1, \cdots, p$. Therefore the posterior distribution of $\theta_1, \theta_2, \cdots, \theta_p$ given $X = x$ are independently gamma distribution, denoted by G$(x_i + k_i, 1 - u)$ for $i = 1, \cdots, p$.

By a result of Zellner(1986), the unique Bayes estimator of θ under the loss function (1.2) is $\delta_B = (\delta_{1B}, \cdots, \delta_{pB}) = -\frac{1}{a}\log(E^{\Theta|X}e^{-a\Theta})$ provided $E^{\Theta|X}e^{-a\Theta}$ is finite. By the above property if

$$1 + (1-u)a > 0 \tag{2.2}$$

then we can obtain the unique Bayes estimator δ_l of θ, relative to the loss function (1.2) and prior distribution (2.1).

That is,

$$\delta_l = \frac{1}{a}\log\{1 + (1-u)a\}(X + k) \tag{2.3}$$

where $k = (k_1, \cdots, k_p)$ with the Bayes risk

$$\begin{aligned} R_l(\delta_l, \Theta) &= \sum_{i=1}^{p} bE\left[e^{a(\delta_{il}-\theta_i)} - a(\delta_{il} - \theta_i) - 1\right] \\ &= \frac{bk}{u}\left[a(1-u) - \log\left\{1 + a(1-u)\right\}\right] \end{aligned} \tag{2.4}$$

where $k = \sum_{i=1}^{p} k_i$ and $E = E^{\Theta}E^{X|\Theta}$ denote a twofold expectation.

Now we consider the estimator $cX + d$, where $c = (c_1, \cdots, c_p)$ and $d = (d_1, \cdots, d_p)$. Then the finite Bayes risk of $cX + d$, relative to the prior distribution (2.1) and the loss function (1.2) is

$$R_l(cX + d, \Theta) = \sum_{i=1}^{p} bE^{\Theta}E^{X|\Theta}\left[e^{a(cX+d-\theta_i)} - a(cX + d - \theta_i) - 1\right]$$

$$= \frac{b}{u}\left[\sum_{i=1}^{p}\left[\frac{\{1+a(1-u)-(1-u)e^{ac}\}}{u}\right]^{-k_i} e^{ad}u \right.$$
$$\left. - a(c-1)(1-u)k - (ad+1)pu\right] \quad (2.5)$$

if $1+a(1-u) > (1-u)e^{ac}$. By the result of Sadooghi-Alvandi(1990). We can show that if $a \le -1$, then $cX+d$ is admissible when $c \ge 0$ and $d \ge 0$, and that if $a > -1$,then $cX+d$ is admissible (inadmissible) when $0 < c < \frac{[\log(1+a)]}{a}$ and $d \ge 0$ $\left(c > \frac{[\log(1+a)]}{a} \text{ or } c = \frac{[\log(1+a)]}{a} \text{ and } d > 0\right)$.

On the other hand, Ghosh(1983) obtain a Bayes estimator δ_s of θ, relative to the loss function (1.1) and the prior distribution (2.1) is

$$\delta_s = (1-u)(X+k-1) \quad (2.6)$$

with Bayes risk

$$R_s(\delta_s, \Theta) = p(1-u) \quad (2.7)$$

where $k_i \ge 1$ for $i = 1, \cdots, p$. Also the Bayes risk of δ_s under the loss function (1.2) is

$$R_l(\delta_s, \Theta) = \frac{b}{u}\left\{\sum_{i=1}^{p}\left[\frac{\{1+a(1-u)-(1-u)e^{a(1-u)}\}}{u}\right]^{-k_i}\right.$$
$$\left. ue^{a(1-u)(k-p)} + au(1-u)p - up\right\} \quad (2.8)$$

Furthermore, we can easily show that if $k_1 = \cdots = k_p = 1$ then δ_l is dominated by δ_s under the loss function (1.2).

3. Emprical Bayes Estimator

In the section 2, we obtained an admissible estimator of θ when u is known. In this section, we shall study the Emprical Bayes estimator of θ when u is unknown. Suppose that u is unknown in this section. Then we estimate u from the data $x = (x_1, \cdots, x_p)$. Under the assumed prior (2.1), $X_1, \cdots, X_p$ are maginally independent such that $X_i's(i = 1, \cdots, p)$ are maginally distributed as negative binomial. Hence the minimal sufficient for u is $S = \sum_{i=1}^{p} X_i$, where maginally S has the negative binomial probability function given by

$$P(S = s) = \binom{s+k-1}{s} u^k (1-u)^s, \quad s = 0, 1, \cdots \tag{3.1}$$

Therefore the minimum variance unbiased estimate (MVUE) of $1-u$ is given by $\frac{S}{(S+k-1)}$.

Substituting this MVUE of $1-u$ into δ_l of (2.3) and δ_s of (2.6) respectively, we obtain emprical Bayes estimatore δ_{Bl} and δ_{Bs} respectively. That is,

$$\delta_{Bl} = \frac{1}{a}\left\{\log\left(1 + \frac{aS}{S+p-1}\right)\right\}(X+1) \tag{3.2}$$

and

$$\delta_{Bs} = \left\{1 - \frac{p-1}{S+p-1}\right\} X \tag{3.3}$$

for the particular case of $k_1 = k_2 = \cdots = k_p = 1$.

First, we compute the risk $r_l(\delta, \Theta)$ and the Bayes risk $R_l(\delta, \Theta)$ of the emprical Bayes estimator δ_{Bl} and δ_{Bs} under the loss function (1.2), respectively.

$$\begin{aligned} r_l(\delta_{Bl}, \Theta) &= b\sum_{i=1}^{p}\left[e^{-a\theta_i} E^X(e^{a\delta_{iBl}}) - a\left\{E^X(\delta_{iBl}) - \theta_i\right\} - 1\right] \\ &= b\sum_{i=1}^{p}\left[e^{-a\theta_i} \sum_{j=0}^{\infty}\sum_{x_i=0}^{j} \frac{\left(\frac{2j+p-1}{j+p-1}\right)^{x_i+1} \binom{j}{x_i}\theta^{x_i}(\bar{\theta}-\theta_i)^{j-x_i} e^{-\bar{\theta}}}{j!}\right. \\ &\quad \left. - a\sum_{j=0}^{\infty} \frac{\left(\frac{\theta_i}{\bar{\theta}} j + 1\right)\log\left(1 + \frac{j}{j+p-1}a\right)\bar{\theta}^j e^{-\bar{\theta}}}{j!} + a\theta_i - 1\right] \end{aligned} \tag{3.4}$$

where $\bar{\theta} = \sum_{i=1}^{p}\theta_i$, and

$$R_l(\delta_{Bl}, \Theta) = bE^X E^{\Theta|X}\left[\sum_{i=1}^{p}\left\{e^{a(\delta_{iBl}-\theta_i)} - a(\delta_{iBl} - \theta_i) - 1\right\}\right]$$

$$= bE^X\left\{\sum_{i=1}^{p}\left[\left(1+\frac{aS}{S+p-1}\right)^{X_i+1}\{1+a(1-u)\}^{-(X_i+1)}\right.\right.$$
$$\left.\left.-(X_i+1)\log\left(1+\frac{aS}{S+p-1}\right)+a(1-u)(X_i+1)-1\right]\right\}.$$

Since $E^X = E^S E^{X|S}$,

$$R_l(\delta_{Bl},\Theta) = b\sum_{s=0}^{\infty}\sum_{i=1}^{p}\sum_{x_i=0}^{s}\left[\left\{\frac{s(a+1)+(p-1)}{(s+p-1)\{1+a(1-u)\}}\right\}^{x_i+1}\right.$$
$$\left.\binom{s+p-x_i-2}{s-x_i}u^p(1-u)^s\right]+b\sum_{s=0}^{\infty}\left[(s+p)\left\{\log\left(1+\frac{as}{s+p-1}\right)\right.\right.$$
$$\left.\left.+a(1-u)\right\}\binom{s+p-1}{s}u^p(1-u)^s\right]-p.$$

Also,

$$r_l(\delta_{Bs},\Theta) = b\sum_{i=1}^{p}\left[e^{-a\theta_i}E^X(e^{a\delta_{iBs}})-a\left\{E^X(\delta_{iBs})-\theta_i\right\}-1\right]$$
$$= b\sum_{i=1}^{p}\left[\frac{e^{-a\theta_i}\sum_{j=0}^{\infty}\left\{(\bar\theta-\theta_i)+\theta_i e^{\frac{j}{j+p-1}a}\right\}^j e^{-\bar\theta}}{j!}\right.$$
$$\left.-a\sum_{j=0}^{\infty}\frac{\left(\frac{j^2}{j+p-1}\right)\theta_i\bar\theta^j e^{-\bar\theta}}{(\bar\theta j!)}+a\theta_i-1\right] \tag{3.5}$$

and

$$R_l(\delta_{Bs},\Theta) = bE^X E^{\Theta|X}\left[\sum_{i=1}^{p}\left\{e^{a(\delta_{iBs}-\theta_i)}-a(\delta_{iBs}-\theta_i)-1\right\}\right]$$
$$= bE^X\left\{\sum_{i=1}^{p}\left[\{1+a(1-u)\}^{-(X_i+1)}e^{\frac{aSX_i}{S+p-1}}-\frac{aSX_i}{S+p-1}\right.\right.$$
$$\left.\left.+a(1-u)(X_i+1)-1\right]\right\}$$
$$= \frac{1}{1+a(1-u)}\sum_{s=0}^{\infty}\sum_{i=1}^{p}\sum_{x_i=0}^{s}\left\{\frac{e^{\frac{-as}{(s+p-1)}}}{1+a(1-u)}\right\}^{x_i}$$
$$-b\sum_{s=0}^{\infty}\left\{\frac{as^2}{s+p-1}+a(1-u)(s+p)\right\}\binom{s+p-1}{s}u^p(1-u)^s-p.$$

Next, computing the $r_s(\delta,\Theta)$ and the Bayes Risk $R_s(\delta,\Theta)$ of δ_{Be} and δ_{Bs} under the loss function (1.1), respectively,

$$r_s(\delta_{Bl},\Theta) = \sum_{i=1}^{p}\theta_i^{-1}\Big[E^X\Big\{\delta_{iBl}^2 - 2\theta_i\delta_{iBl} + \theta_i^2\Big\}\Big]$$

$$= \sum_{i=1}^{p}\theta_i^{-1}\Big[\sum_{j=0}^{\infty}\Big\{\frac{1}{j!}\Big(\frac{\log\Big(1+\frac{j}{j+p-1}a\Big)}{a}\Big)^2$$

$$\Big(j^2\Big(\frac{\theta_i}{\bar{\theta}}\Big)^2 + 3j\frac{\theta_i}{\bar{\theta}} - j\Big(\frac{\theta_i}{\bar{\theta}}\Big)^2 + 1\Big)\bar{\theta}^j e^{-\bar{\theta}}\Big\}$$

$$- 2\theta_i\sum_{j=0}^{\infty}\Big\{\frac{1}{a}\log\Big(1+\frac{j}{j+p-1}a\Big)\Big(\frac{j\theta_i}{\bar{\theta}}+1\Big)\Big\}\frac{\bar{\theta}^j e^{-\bar{\theta}}}{j!} + \theta_i^2\Big] \quad (3.6)$$

and

$$R_s(\delta_{Bl},\Theta) = E^X E^{\Theta|X}\Bigg[\sum_{i=1}^{p}\frac{\Big\{\Big(\frac{X_i+1}{a}\Big)\log\Big(1+\frac{aS}{S+p-1}\Big)-\theta_i\Big\}^2}{\theta_i}\Bigg].$$

Since

$$E(\theta_i|X_i = x_i) = (1-u)(x_i+1)$$

$$E(\theta_i^{-1}|X_i = x_i) = (1-u)^{-1}x_i^{-1} \quad \text{if} \quad x_i \geq 1,$$

$$R_s(\delta_{Bl},\Theta) = \sum_{s=0}^{\infty}\Big\{\Big[\log\Big(1+\frac{as}{s+p-1}\Big)\Big]^2\Big(\frac{s}{a^2}+2p\Big)(1-u)^{-1}$$

$$+\Big[\log\Big(1+\frac{as}{s+p-1}\Big)\Big]^2\Big(\frac{2s+2p}{a}\Big) + (1-u)(s+p)\Big\}$$

$$\binom{s+p-1}{s}u^p(1-p)^s + \sum_{s=0}^{\infty}\sum_{i=1}^{p}\sum_{x_i=1}^{s}\Big\{\log\Big(1+\frac{as}{s+p-1}\Big)\Big\}^2$$

$$\binom{s+p-x_i-2}{p-2}u^p(1-u)^s$$

Also, by the results of Ghosh(1983).

$$r_s(\delta_{Bs},\Theta) = \sum_{i=1}^{p}\theta_i^{-1}E^X(\delta_{iBs}-\theta_i)^2$$

$$= \sum_{i=1}^{p} \theta_i^{-1} \left[\sum_{j=0}^{\infty} \left\{ \left(\frac{j}{j+p-1} \right)^2 j^2 \left(\frac{\theta_i}{\bar{\theta}} \right)^2 + j \left(\frac{\theta_i}{\bar{\theta}} - \left(\frac{\theta_i}{\bar{\theta}} \right)^2 \right) \frac{\bar{\theta}^j e^{-\bar{\theta}}}{j!} \right\} \right.$$

$$\left. - 2 \frac{\theta_i^2}{\bar{\theta}} \sum_{j=0}^{\infty} \left\{ \frac{(j^2 \bar{\theta}^j e^{-\bar{\theta}})}{[(j+p-1)j!]} \right\} + \theta_i^2 \right] \tag{3.7}$$

and

$$R_s(\delta_{Bs}, \Theta) = E^X E^{(\Theta|X)} \left[\sum_{i=1}^{p} \frac{\left\{ \frac{aSX_i}{S+p-1} - \theta_i \right\}^2}{\theta_i} \right]$$

$$= \sum_{s=0}^{\infty} \left\{ \left(\frac{as}{s+p-1} \right)^2 s(1-u)^{-1} - \frac{2aps}{s+p-1} + (1-u)(s+p) \right\}$$

$$\binom{s+p-1}{s} u^p (1-u)^s.$$

4. Monte Carlo Simulation Result

A some Monte Carlo Simulation were carried out to compare the risks under a LINEX loss function (1.2) and squared error loss function (1.1) for some given values of θ and a, b.

The Poisson random variate were generated by the method of Kenep and Kenep (1991). And numbers of replication in simulation were doing 100 times on a 386PC with math-coprocessor. From the table, we can know the following: δ_{Bs} is better than δ_{Bl} under weighted squared loss function, and δ_{Bl} is better than δ_{Bs} under a LINEX loss function. However, the difference of $r_s(\delta_{Bl}, \Theta)$ and $r_l(\delta_{Bs}, \Theta)$ is larger than difference of $r_l(\delta_{Bs}, \Theta)$ and $r_l(\delta_{Bl}, \Theta)$, as p is larger and larger and also a is larger and larger. Hence, for the emprical Bayes estimator of θ, δ_{Bl} is useful than δ_{Bs}.

References

1. Ghosh, M. (1983), Estimates of multiple poisson means; Bayes and empirical Bayes, *Statistics and Decisions*, 1, 183-195.

2. Kenep, C. D. and Kenep, A. W. (1991), Poisson random variate Generation, *Applied Statistics*, 40. NO 1, 143-158.

3. Sadooghi-Alvandi, S. M. (1990), Estimation of the parameter of a Poisson distribution using a LINEX loss function, *Austrial. J. Statist.*, 32(20), 393-398.

4. Tsui, K. W. and Press, S. J. (1977), Simutaneous estimation of several Poisson parameters under k-normalized squard error loss, *Ann. of Statist.*, 10(1), 93-100.

5. Varian, H. R. (1975), A Bayesian approach to real estate assessment, *Studies in Bayesian Econometrics and Statistics* in Honour of Leonard J. Savage. (eds. S. E. Frienberg and A. Zellner), 195-208, North-Holland, Amsterdam.

6. Zellner, A. (1986), Bayesian estimation and prediction using asymmetric loss function, *J. Amer. Statist. Assoc.* 81, 446-451.