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On the Weak Law of Large Numbers for the Sums of Sign-Invariant Random Variables

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ABSTRACT

We consider various types of weak convergence for sums of sign–invariant random variables. Some results show a similarity between independence and sign–invariance. As a special case, we obtain a result which strengthens a weak law proved by Rosalsky and Teicher [6] in that some assumptions are deleted.

1. Introduction

The concept of sign–invariant random variables was introduced by S.M.Berman [1]: $Y_1, \cdots, Y_k$ are called sign–invariant if the 2^k joint distribution corresponding to the sets $(\varepsilon_1 Y_1, \cdots, \varepsilon_k Y_k), \varepsilon_1 = \pm 1, \cdots, \varepsilon_k = \pm 1$ are all the same. A family of random variables $\{Y_t, t \in T\}$, where T is some index set, is called sign–invariant if every finite subfamily consists of sign–invariant random variables. It is obvious that a sequence of independent random variables with a common distribution function $F(x)$ is sign–invariant if and only if $F(x)$ is symmetric,i.e.,every one–dimensional distribution function is invariant under changes in signs. The sign–invariance of exchangable random variables is discussed in [2]. Basic properties of sign–invariant random variables are given in [3]. A fundamental property is that sign–invariant

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random variables are conditionally independent given their absolute values (see Berman [3, Lemma 1.2]). The sign–invariance property is also very close to a martingale property (see Berman [3, Corollary 1.1]).

In this paper, we consider various types of weak convergence theorem for sums of sign–invariant random variables. In section 2, we introduce sum basic lemmas which are important tools throughtout this paper. Lemma 3 (Berman [3, Lemma 1.2]) is useful to construct sign–invariant random variables. In section 3, we give sufficient conditions for the convergence of sums of sign–invariant random variables and extend this to the case of random indices. However, unlike the independent case, the converse is false. We give an example for this and we also construct a sequence of identically distributed sign–invariant random variables which is not recurrent. In section 4, we study the weak convergence of weighted sums of sign–invariant random variables. Theorem 5 strengthens somewhat Theorem 2 of Rosalsky and Teicher [6] in that the assumption that Y_1 is unbounded or $\sigma_n^2 = o(s_n^2)$ can be deleted. The proof of Theorem 5 parallels that of Rosalsky and Teicher [6] but is much simple. Thorem 6 shows a similarity between independence and sign–invariance like Theorem 1.1 of Berman [3].

2. Basic properties of sign–invariant families

Lemma 1. If $\{X_n, n \geq 1\}$ is a sequence of sign-invariant random variables then for every $a > 0$, $\{X_n I_{\{|X_n| \leq a\}}, \ n \geq 1\}$ is sign-invariant and hence each mean zero and median zero, where $I\{\cdot\}$ denotes an indicator of $\{\cdot\}$.

Proof.

$$\begin{aligned}
&P\{\epsilon_1 X_1 I\{|X_1| \leq a\} \in A_1, \cdots, \epsilon_n X_n I\{|X_n| \leq a\} \in A_n\} \\
&= P\{\epsilon_1 X_1 \in A_1 \cap [-a, a], \cdots, \epsilon_n X_n \in A_n \cap [-a, a]\} \\
&= P\{X_1 \in A_1 \cap [-a, a], \cdots, X_n \in A_n \cap [-a, a]\} \\
&= P\{X_1 I\{|X_1| \leq a\} \in A_1, \cdots, X_n I\{|X_n| \leq a\} \in A_n\}.
\end{aligned}$$

If $X_1, X_2, \cdots$ are sign–invariant with $E|X_n| < \infty, \ n \geq 1,$ then $X_1, X_1 + X_2, \cdots$ form a martingale (see Berman [3, Corollary 1.1]). Hence the following

Lemma 2 is immediate from martingale generalization of Kolmogorov's inequality (see Chow–Teicher [4, Theorem 7.4.8]).

Lemma 2. If $\{X_j, 1 \le j \le n\}$ are sign–invariant random variables with $EX_j^2 < \infty$, $S_j = \sum_{i=1}^{j} X_i$, $1 \le j \le n$, then for every $\epsilon > 0$

$$P\{\max_{1\le j\le n} |S_j| \ge \varepsilon\} \le \frac{1}{\varepsilon^2} \sum_{j=1}^{n} EX_j^2.$$

The following lemma of Berman [3, Lemma 1.2] is useful.

Lemma 3. $X_1, \cdots, X_k$ are sign–invariant if and only if they are conditionally independent, given $|X_1|, \cdots, |X_k|$, with the conditional joint characteristic function $\Pi_{j=1}^{k} \cos u_j X_j$.

3. Sums of sign–invariant random variables

Theorem 1. If $\{X_n, n \ge 1\}$ are sign-invariant and identically distributed random variables obeying

$$nP\{|X_1|^p > n\} \longrightarrow 0 \tag{2.1}$$

for some $0 < p < 2$, then, setting $S_n = \sum_{i=1}^{n} X_i$,

$$S_n/n^{1/p} \xrightarrow{P} 0,$$

where $\xrightarrow{P}$ stands for convergence in probability.

Proof. Set $X_j' = X_j I\{|X_j|^p \le n\}$ for $1 \le j \le n$ and $S_n' = \sum_{j=1}^{n} X_j'$. Then, for each $n \ge 2$ and for $\epsilon > 0$, $P\{|(S_n/n^{1/p}) - (S_n'/n^{1/p})| \ge \epsilon\} \le P\{S_n \ne S_n'\} = P\{\cup_{j=1}^{n}[X_j \ne X_j']\} \le nP\{|X_1|^p > n\}$ so that (2.1) entails $(S_n'/n^{1/p}) - (S_n/n^{1/p}) \xrightarrow{P} 0$. Thus, to prove the theorem it suffices to verify that

$$S_n'/n^{1/p} \xrightarrow{P} 0. \tag{2.2}$$

Since by Lemma 1 and Corollary 1.1 of Berman [3], $\{X_i', 1 \leq i \leq n\}$ form a martingale difference sequence, i.e., $\{\sum_{j=1}^{i} X_j' = S_i',\ 1 \leq i \leq n\}$ form a martingale , and hence are orthogonal elements of L^2, we have, by Lemma 5.1.1 (4) of Chow and Teicher [4] and (2.1),

$$\begin{aligned}
E(S_n')^2 = nE(X_1')^2 &= n\sum_{j=1}^{n}\int_{\{j-1<|X_1|^p\leq j\}} X_1^2 dP \\
&\leq n\sum_{j=1}^{n} j^{2/p}[P\{|X_1|^p > j-1\} - P\{|X_1|^p > j\}] \\
&= n[P\{|X_1|^p > 0\} - n^{2/p}P\{|X_1|^p > n\} \\
&\quad + \sum_{j=1}^{n-1}((j+1)^{2/p} - j^{2/p})P\{|X_1|^p > j\}] \\
&\leq n[1 + c\sum_{j=1}^{n-1}((j+1)^{\frac{2}{p}-1} - j^{\frac{2}{p}-1})jP\{|X_1|^p > j\}], \qquad (2.3)
\end{aligned}$$

where c is an unimportant constant. By the hypothesis (2.1), $jP\{|X_1|^p > j\}$ goes to zero as $j \to \infty$ and $\sum_{j=1}^{n}((j+1)^{\frac{2}{p}-1} - j^{\frac{2}{p}-1}) = (n+1)^{\frac{2}{p}-1} - 1$. Thus, by Toeplitz Lemma (see Ash [1, p270]),

$$E(S_n')^2 = o(n^{2/p}),$$

which implies (2.2) and hence completes the proof.

Remark 1. The converse of Theorem 1 also holds if we assume further that the random variables X_n are independent (see Chow and Teicher [4, Theorem 5.2.4]). But in general the converse of Theorem 2.1 does not hold. For this we consider the following example.

Example 1. By Lemma 3, we note that for a sequence of sign–invariant random variables $\{X_n, n \geq 1\}$, the X's are conditionally independent given $|X_1|, |X_2|, \cdots$ and each X_n assumes the values $\pm|X_n|$ with probability $1/2, 1/2$. First we find a sequence of positive constants $\{a_n, n \geq 1\}$ such that $\sum_{i=1}^{\infty} a_i < \infty$ and $n\sum_{k>n} a_k \not\to 0$. Let $P\{(|X_1|, |X_2|, \cdots) = (n, n, \cdots)\} =$

$ca_n, n \geq 1$, where $c = (\sum_{i=1}^{\infty} a_i)^{-1}$. Then

$$nP\{|X_1| > n\} = n \sum_{k>n} ca_k \not\longrightarrow 0.$$

Thus, the condition (2.1) is not satisfied. Now for given $(|X_1|, |X_2|, \cdots) = (n, n, \cdots)$, $\{X_n, n \geq 1\}$ are independent and each X_n assumes the value $\pm n$ with probability $(1/2, 1/2)$. Hence by the classical strong law of large numbers, $P\{lim S_n/n = 0 | |X_1| = |X_2| = \cdots = n\} = 1$, $n \geq 1$. Then $P\{lim S_n/n = 0\} = E[P\{lim\ S_n/n = 0 | |X_1|, |X_2|, \cdots\}] = 1$. It follows that $S_n/n \xrightarrow{\text{P}} 0$.

The versatility of the Lemma 2 will be exemplified extending Theorem 1 to the case of random indices.

Theorem 2. If $\{X_n, n \geq 1\}$ are sign-invariant and identically distributed random variables obeying

$$nP\{|X_1|^p > n\} = o(1) \tag{2.4}$$

for some $1 \leq p < 2$ and $\{T_n, n \geq 1\}$ are positive integer-valued random variables satisfying

$$\frac{T_n}{n^{\frac{1}{p}}} \xrightarrow{\text{P}} c, \text{ where } 0 < c < \infty. \tag{2.5}$$

Then, setting $S_n = \sum_{i=1}^{n} X_i$,

$$S_{T_n}/T_n \xrightarrow{\text{P}} 0.$$

Proof. Define $X'_j = X_j I\{|X_j|^p \leq n\}$, $S'_j = \sum_{i=1}^{j} X'_i$, and $c_n = [c \cdot n^{\frac{1}{p}}] =$ the largest integer $\leq c \cdot n^{\frac{1}{p}}$. Then

$$P\{S_{T_n} \neq S'_{T_n}, T_n \leq 2c_n\} \leq P\{\cup_1^{2c_n}[|X_j|^p > n]\}$$

$$\leq \sum_{j=1}^{2c_n} P\{|X_j|^p > n\} \leq 2c_n P\{|X_1|^p > n\} = o(1)$$

by (2.4), hence, by (2.5)

$$P\{S_{T_n} \neq S'_{T_n}\} \leq P\{S_{T_n} \neq S'_{T_n}, T_n \leq 2c_n\} + P\{T_n > 2c_n\} = o(1),$$

that is,

$$S_{T_n} - S'_{T_n} \xrightarrow{\text{P}} 0. \tag{2.6}$$

If we define $B_j = \{|S'_j| > n^{\frac{1}{p}}\epsilon\}$ and $D_n = \cup_{j=1}^{2c_n} B_j$, observing that $E(X'_1)^2 = o(n^{\frac{2}{p}-1})$ in (2.3), by Lemma 2

$$P\{D_n\} \leq \frac{1}{n^{\frac{2}{p}}\epsilon^2}\sum_{j=1}^{2c_n} E(X'_j)^2 \leq \frac{2c}{n^{\frac{1}{p}}\epsilon^2} E(X'_1)^2 = o(1).$$

Thus,

$$\begin{aligned} P\{B_{T_n}\} &\leq P\{T_n \leq 2c_n, B_{T_n}\} + P\{T_n > 2c_n\} \\ &\leq P\{D_n\} + P\{T_n > 2c_n\} = o(1), \end{aligned}$$

so that

$$\frac{S'_{T_n}}{n^{\frac{1}{p}}} \xrightarrow{\text{P}} 0.$$

Hence, by (2.6) and (2.5)

$$\frac{S_{T_n}}{T_n} = \frac{n^{\frac{1}{p}}}{T_n}\frac{S_{T_n}}{n^{\frac{1}{p}}} \xrightarrow{\text{P}} 0.$$

Chung and Ornstein (see Chow and Teicher [4, Example 5.2.1]) showed that if $\{X_n, n \geq 1\}$ are symmetric i.i.d. with $nP\{|X_1| > n\} = o(1)$ then the random walk $\{S_n = \sum_{i=1}^n X_i, n \geq 1\}$ is recurrent. But if $\{X_n, n \geq 1\}$ are sign–invariant and identically distributed random variables with $nP\{|X_1| > n\} = o(1)$, then it is not true. See the following example.

Example 2. Let $P\{|X_1| = 1, (|X_2|, |X_3|, \cdots) = (2, 2, \cdots)\} = \frac{1}{2} = P\{|X_1| = 2, (|X_2|, |X_3|, \cdots) = (1, 1, \cdots)\}$. Then we easily check that $P\{|S_n| < \epsilon \ i.o. || X_1| = 1, (|X_2|, |X_3|, \cdots) = (2, 2, \cdots)\} = 0$ for $0 < \epsilon < 1$. Hence $\{S_n, n \geq 1\}$ is not recurrent.

The following two results whose proofs, using orthogonality in L^2, are similar to those of Chow and Teicher [4, Theorem 10.1.1 and Corollary 10.1.3], are stated without proof.

Theorem 3. Let $\{X_{nj}, 1 \le j \le k_n \to \infty, n \ge 1\}$ be a double array of rowwise sign-invariant random variables obeying

$$\sum_{j=1}^{k_n} P\{|X_{nj}| \ge \varepsilon\} \to 0, \quad \varepsilon > 0,$$

$$\sum_{j=1}^{k_n} EX_{nj}^2 I_{\{|X_{nj}|<1\}} \to 0.$$

Then we have

$$S_n \xrightarrow{\text{P}} 0.$$

Corollary 1. Let $\{X_n, , n \ge 1\}$ be sign-invariant random variables, $S_n = \sum_{j=1}^n X_j$, and $\{b_n, n \ge 1\}$ be constants with $0 < b_n \to \infty$. If

$$\sum_{j=1}^{n} P\{|X_j| \ge b_n\} = o(1),$$

$$\frac{1}{b_n^2} \sum_{j=1}^{n} EX_j^2 I_{\{|X_j|<b_n\}} = o(1),$$

then we have

$$S_n/b_n \xrightarrow{\text{P}} 0.$$

Remark 2. Example 1 also shows that the converse of above results does not hold.

4. Sums of weighted sign–invariant random variables

Rosalsky and Teicher [6, Theorem 2] proved a weak law for weighted symmetric i.i.d. random variables. The next theorem is similar to that of Rosalsky and Teicher

[6, Theorem 2]. Using this result we generalize Rosalsky and Teicher [6, Theorem 2].

Theorem 4. Let $S_n = \sum_{i=1}^n \sigma_i Y_i$ and $s_n{}^2 = \sum_{j=1}^n \sigma_j^2, n \geq 1$, where $\{\sigma_n, n \geq 1\}$ are constants and $\{Y_n, n \geq 1\}$ is a sign-invariant, stationary and ergodic process with $EY_1^2 < \infty$. If for some sequence of positive constants a_n,

$$\limsup_{n\to\infty} S_n/a_n = 1, \quad a.s. \tag{2.7}$$

then the weak law of large numbers

$$S_n/a_n \xrightarrow{\text{P}} 0 \tag{2.8}$$

obtains.

To prove the theorem the following lemma is used.

Lemma 4. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $P\{X_n = \pm x_n\} = \frac{1}{2}, n \geq 1$, $S_n = \sum_{j=1}^n X_j$ and $s_n^2 = \sum_{j=1}^n x_j{}^2 = Var(S_n)$ such that $x_n{}^2 = o(s_n^2) \to \infty$. If for some sequence of positive constants a_n,

$$\limsup\nolimits_{n\to\infty} S_n/a_n = 1 \quad a.s. \tag{2.9}$$

then

$$S_n/a_n \xrightarrow{\text{P}} 0.$$

Proof. The conditions in this lemma entail the classical Lindeberg condition for asymptotic normality of S_n/s_n and hence $S_{n(k)}/s_{n(k)} \xrightarrow{\text{d}} N(0,1)$ as $k \to \infty$ for any subsequence $n(k) \to \infty$, where $\xrightarrow{\text{d}}$ stands for convergence in distribution. Now $s_n = o(a_n)$ since otherwise $s_{n(k)} > \alpha a_{n(k)}$ for some subsequence $n(k) \to \infty$, for some $\alpha > 0$, implying for $\epsilon > 0$,

$$\lim\ P\{S_{n(k)} \leq (1+\epsilon)a_{n(k)}\} \leq \lim\ P\{S_{n(k)} \leq (1+\epsilon)\frac{1}{\alpha}s_{n(k)}\}$$

$$= \lim\ P\{\frac{S_{n(k)}}{s_{n(k)}} \leq (1+\epsilon)\frac{1}{\alpha}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(1+\epsilon)\frac{1}{\alpha}} e^{-x^2/2}dx < 1,$$

which contradicts (2.9). Thus, $s_n = o(a_n)$, which implies

$$P\{|\frac{S_n}{a_n}| > \epsilon\} = P\{|\frac{S_n}{s_n}| > \frac{\epsilon a_n}{s_n}\} \longrightarrow 0 \quad as\ n \to \infty,$$

and this completes the lemma.

Proof of Theorem 4. By the Birkhoff's ergodic theorem $Y_1^2 + \cdots + Y_n^2/n \to EY_1^2$ $a.s.$ and hence

$$Y_n^2 = o(Y_1^2 + \cdot + Y_n^2) \quad a.s. \tag{2.10}$$

It will now be shown that

$$s_n^2 \longrightarrow \infty. \tag{2.11}$$

Now, suppose s_n^2 converges. Since $\{\sum_{k=1}^n \sigma_k Y_k, n \geq 1\}$ form a martingale and $0 < \sum_{n=1}^\infty E(\sigma_n Y_n)^2 < \infty$, the martingale convergence theorem (see Chow and Teicher [4, Theorem 7.4.3]) ensures that S_n converges a.s. to a non-degenerate random variable S with $ES = 0$, which contradicts (2.7) since $P\{S < 0\} > 0$. Thus, by (2.10) and (2.11) we have that $\sigma_1^2 X_1^2 + \cdots + \sigma_n^2 X_n^2 \to \infty$ and $\sigma_n X_n^2 = o(\sigma_1^2 X_1^2 + \cdots + \sigma_n^2 X_n^2)$, and hence by Lemma 3 and 4, for $\epsilon > 0$, $P\{|S_n/a_n| > \epsilon | |X_1|, |X_2|, \cdots\} \longrightarrow 0$ $a.s.$ Thus, we obtain (2.8) by the dominated convergence theorem.

Combining Theorem 4 with the unbounded case of Y_1 in Rosalsky and Teicher [6, Theorem 2], we have the following theorem which strengthens somewhat Theorem 2 of Rosalsky and Teicher [6] itself.

Theorem 5. Let $S_n = \sum_{j=1}^n \sigma_i Y_i$ and $s_n^2 = \sum_{j=1}^n \sigma_j^2, n \geq 1$, where $\{\sigma_n, n \geq 1\}$ are constants (not necessary nonzero) and $\{Y_n, n \geq 1\}$ are symmetric i.i.d. random variables. If for some sequence of positive constants a_n,

$$\limsup_{n\to\infty} S_n/a_n = 1, \text{ a.s.}$$

then the weak law of large numbers

$$S_n/a_n \xrightarrow{\text{P}} 0$$

obtains.

Theorem 6. Let $\{X_n, n \geq 1\}$ be a sequence of sign–invariant, identically distributed random variables. Let $A = \{a_{nk} : n \geq 1, k \geq 1\}$ be a double sequence of real numbers which satisfy

$$a) \quad \lim_{n\to\infty} a_{nk} = 0 \text{ for every } k,$$

$$b) \quad \sum_{k=1}^{\infty} |a_{nk}|^r \leq C \text{ for all } n,$$

where $0 < r \leq 1$ and C is some positive constant.

Write $S_n = \sum_{k=1}^{\infty} a_{nk}X_k$. Then

$$S_n \xrightarrow{\text{P}} 0 \text{ if and only if } \max_k |a_{nk}| \to 0 \text{ as } n \to \infty.$$

Proof. For the sufficiency, just follow the proof of Rohatgi [5, Theorem 1] using orthogonality in L^2. For the necessity, let $g(u) = E|\cos uX_k|$. Since $S_n \xrightarrow{\text{P}} 0$, we have $Ee^{ius_n} \to 1$ as $n \to \infty$. But noting that by Lemma 4

$$Ee^{ius_n} = E[E[\Pi_{k=1}^{\infty} \cos(ua_{nk}X_k)||X_1|, |X_2|, \cdots]],$$

$$|Ee^{ius_n}| \leq g(a_{nm}u) \leq 1$$

for any m, so that for any sequence k_n,

$$g(a_{nk_n}u) \longrightarrow 1. \tag{2.12}$$

Since X_k is non–degenerate, and g is a continuous function, there exists a u_0 such that $|g(u)| < 1$ for $0 < |u| < u_0$. Letting $u = u_0/2C^{\frac{1}{r}}$, it follows that

$$|a_{nk_n}| \leq C^{\frac{1}{r}}u = u_0/2$$

and then $a_{nk_n}u \to 0$ by (2.10). Choosing k_n to satisfy $|a_{nk_n}| = \max_k|a_{nk}|$ completes the proof.

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