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Multivariate EWMA Control Charts for the Variance-Covariance Matrix with Variable Sampling Intervals

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ABSTRACT

Multivariate exponentially weighted moving average (EWMA) control charts for monitoring the variance-covariance matrix are investigated. A variable sampling interval (VSI) feature is considered in these charts. Multivariate EWMA control charts for monitoring the variance-covariance matrix are compared on the basis of their average time to signal (ATS) performances. The numerical results show that multivariate VSI EWMA control charts are more efficient than corrsponding multivariate fixed sampling interval (FSI) EWMA control charts.

1. Introduction

Control charts are used to monitor quality variables from a process to detect changes in the parameters of the distribution of these variables. A control chart is maintained by taking samples from a process and plotting the relevant statistic computed from the samples in time order on the control chart.

When control charts are used to monitor production process the main objective is to detect any change in the process that may affect the quality of the output of

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the process. The usual practice when monitoring a control chart is to take samples from the process at fixed length sampling intervals.

Reynolds and Arnold (1989), Reynolds (1989), and Reynolds et al. (1988) investigated the properties of control charts in which the sampling interval is varied as a fuction of what is observed from the process. The basic idea of a variable sampling interval (VSI) control chart is that after a sample is taken, the time interval until the next sample should be short if there is some indication of a process change and long if there is no indication of a process change. If the indication of a process change is strong enough then the VSI chart signals in the same way as a fixed sampling interval (FSI) chart.

Reynolds and Arnold (1989) and Reynolds (1989) showed that the optimal VSI chart uses only the shortest possible interval and longest possible interval from a range of possible sample intervals. A chart is said to be optimal if it minimizes the time required to detect a shift in a process parameter subject to a given false alarm rate and a given average sampling rate.

There are many situations in which the simultaneous control of two or more quality characteristics is necessary. The original work in multivariate quality control was introduced by Hotelling (1947). Alt (1984) and Jackson (1985) reviewed much of the literature on the multivariate control charts. The multivariate control charts using the VSI idea have been studied by Hui et al.(1980) and Chengalur-Smith et al.(1993) but this work was only for the Shewhart control charts. We will be looking at multivariate EWMA charts for monitoring the variance-covariance matrix with variable sampling intervals.

Suppose that the process of interest has p quality characteristics represented by the random vector $\underline{X} = (X_1, X_2, ..., X_p), p = 2, 3, ...$, and $\underline{X}$ has a multivariate normal distribution with mean vectors $\underline{\mu}_i, i = 1, 2, ...$, and variance-covariance matrix Σ. Let the sample of n observations taken at the sampling point be represented by $np \times 1$ vector $\underline{X_i} = (\underline{X}_{i1}', \underline{X}_{i2}', ..., \underline{X}_{in}')$, where $\underline{X}_{ij}', = (X_{ij1}, X_{ij2}, ..., X_{ijn})$ is the j^{th} observation vector among n observation vectors taken at the i^{th} sampling point. It will be assumed that the observation vectors within and between samples are independent. Even though most control charts make this assumption, one should note that this is perhaps not very realistic, because production processes are inherently time dependent.

Suppose that the objective is to monitor Σ where the target value Σ_0 is known. We will consider the case in which the primary objective is to detect changes in the variances, not in the correlation coefficients. Several different control statistics for Σ will be presented since different statistics can be used to describe variability. In the univariate case, the S^2-chart is used to control the variance under the normality assumption. The S^2 -chart signals for large values of S^2 or equivalently for large values of $V_i = (n-1){S_i}^2/{\sigma_0}^2$, where ${S_i}^2 = \sum_{j=1}^{n} \left(X_{ij} - \overline{X}_i\right)^2/(n-1)$. One possible multivariate version of V_i is

$$V_i = \sum_{j=1}^{n} \left(\underline{X}_{ij} - \overline{\underline{X}}_i\right)' {\Sigma_0}^{-1} \left(\underline{X}_{ij} - \overline{\underline{X}}_i\right),$$

where $A_i = \sum_{j=1}^{n} \left(\underline{X}_{ij} - \overline{\underline{X}}_i\right)\left(\underline{X}_{ij} - \overline{\underline{X}}_i\right)'$. When $\Sigma = \Sigma_0$, V_i has a chi-squared distribution with (n-1) p degrees of freedom. Hoteling (1947) proposed the use of the Lawley-Hotelling statistic V_i in monitoring the process variance-covariance matrix. The distribution of V_i was studied by Lawley (1938) and Hotelling (1951).

Hui (1980) studied the use of the sample generalized variance in monitoring the process variance-covariance matrix using a statistic $L_i = \mid A_i/(n-1) \mid / \mid \Sigma_0 \mid$. It is known that $(n-1)^{1/2}(L_i - 1)$ is asymptotically normal with mean 0 and variance $2p$ (Anderson (1958)). Another chart can be constructed by using the likelihood ratio statistic for testing $H_0 : \Sigma = \Sigma_0$ $vs.$ $H_1 : \Sigma \neq \Sigma_0$.

In general, if the process shifts from Σ_0 to Σ_1 then it is difficult to obtain the distribution of V_i. Thus, in order to evaluate the properties of the charts for Σ it is necessary to carry out computer simulations.

2. Multivariate EWMA Charts for the Variance-Covariance Matrix

We propose a multivariate EWMA control chart based on the "accumulate-combine approach " that accumulates past sample information for each parameter and then combine the separate accumulations into a univariate statistic. The multivariate control chart based on the accumulate-combine approach is more efficient in terms of ATS than the multivariate control chart based on "combine-accumulate approach " that combines the multivariate data into a univariate statistic and then accumulate over past samples. Multivariate EWMA control charts based on the accumulate-combine approach can be constructed by forming a univariate statistic from vectors of EWMA's.

In the univariate case, an EWMA control chart for σ^2 can be constructed by using the statistic

$$Y_k = (1-\lambda)Y_{k-1} + \lambda \sum_{j=1}^{n} \left(\frac{X_{kj} - \overline{X}_{k.}}{\sigma_0} \right)^2 . \tag{2.1}$$

By repeated substitution in (1), it can be shown that

$$Y_k = (1-\lambda)^k Y_0 + \sum_{i=1}^{k} \lambda(1-\lambda)^{k-i} \left\{ \sum_{j=1}^{n} \left(\frac{X_{ij} - \overline{X}_{i.}}{\sigma_0} \right)^2 \right\}, \tag{2.2}$$

$k = 1, 2, \cdots$ and $0 < \lambda \leq 1$.

In the multivariate case, define vectors of EWMA's $\underline{Y_k} = (Y_{k1}, Y_{k2}, \cdots, Y_{kp})'$, where

$$Y_{kl} = (1-\lambda)^k Y_{l0} + \sum_{i=1}^{k} \lambda(1-\lambda)^{k-i} \left\{ \sum_{j=1}^{n} \left(\frac{X_{ijl} - \overline{X}_{i.l}}{\sigma_{0l}} \right)^2 - (n-1) \right\}, \tag{2.3}$$

$k = 1, 2, \cdots$, $Y_{lo} = 0$, and $0 < \lambda_l \leq 1$, $l = 1, 2, \cdots, p$. A multivariate EWMA chart for Σ is based on the statistic

$$T_k{}^2 = \underline{Y_k}' \Sigma_{\underline{Y_k}}{}^{-1} \underline{Y_k} \,. \tag{2.4}$$

where $\Sigma_{\underline{Y_k}}$ is the variance-covariance matrix of $\underline{Y_k}$ which will be given in Theorem 1. The Vector $\underline{Y_k}$ is one possible multivariate extension of Y_k in (2.2). In general, it is difficult to obtain the distribution of $\underline{Y_k}$, but the asymptotic distribution will be obtained for the case in which the process is in control. To simplify notations, let

$$Z_{il} = \sum_{j=1}^{n} \left(\frac{X_{ijl} - \overline{X}_{i.l}}{\sigma_{0l}} \right)^2 - (n-1), \quad \text{and} \quad \underline{Z_i} = (Z_{i1}, Z_{i2}, \cdots, Z_{ip})',$$

for $l = 1,2,\cdots,p$, $i = 1,2,\cdots$, then the multivariate EWMA vectors can be expressed as

$$\underline{Y_k} = (1-\lambda)\underline{Y}_{k-1} + \lambda \underline{Z_k} \,. \tag{2.5}$$

By repeated substitution in (2.5), it can be shown that

$$\underline{Y_k} = \sum_{i=1}^{k} \lambda(1-\lambda)^{k-i} \underline{Z_i} \,.$$

It is easy to show that

$$E(Z_{il} \mid \mu_l = \mu_{0l}, \sigma_l = \sigma_{il}) = (n-1)\left[\left(\frac{\sigma_{il}}{\sigma_{0l}}\right)^2 - 1\right], \quad l = 1,2,\cdots,p.$$

Thus, under the assumption that $\underline{\mu} = \underline{\mu_0}$ and $\Sigma = \Sigma_1$, the expected value of the random vector $\underline{Z_i}$, denoted by $\underline{\mu}_{\underline{z}}$, is

$$\underline{\mu}_{\underline{z}} = (n-1)\left[(\frac{\sigma_{11}}{\sigma_{01}})^2 - 1,\ (\frac{\sigma_{12}}{\sigma_{02}})^2 - 1,\ (\frac{\sigma_{1p}}{\sigma_{0p}})^2 - 1\right]' .$$

If $\Sigma = \Sigma_0$, then $\underline{\mu}_{\underline{z}} = 0$.

Theorem 1. The variance-covariance matrix of $\underline{Y_k}$ when a process is in control and $\underline{Y_0} = \underline{0}$ is

$$\Sigma_{\underline{Y_k}} = \frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2k}\right]\Sigma_{\underline{Z}}\,, \quad \Sigma_{\underline{Z}} = 2(n-1)R^2,$$

where R^2 is used to denote the matrix whose $(l,l')^{th}$ element is the 2^{nd} power of $(l,l')^{th}$ component of R which is the correlation matrix of $\underline{X} = (X_1, X_2, ..., X_p)$.

Proof. It is easy to show that from the fact $\underline{Y_k} = \sum_{i=1}^{k} \lambda(1-\lambda)^{k-i}\underline{Z_i}$,

$$\Sigma_{\underline{Y_k}} = \sum_{i=1}^{k} Cov\left[\lambda(1-\lambda)^{k-i}\underline{Z_i}\right] = \frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2k}\right]\Sigma_{\underline{Z}}\,.$$

When a process is in control, the mean vector and variance-covariance matrix of $\underline{Z_i}$ is defined as follows: recall that $Z_{il} + (n-1) = \sum_{j=1}^{n}\left(\frac{X_{ijl} - \overline{X}_{i.l}}{\sigma_{0l}}\right)^2$ has a chi-squared distribution with (n-1) degrees of freedom. Thus $E(Z_{il})=0$ and $Var(Z_{il})=2(n-1)$ for $l=1,2,\cdots,p$, $i=1,2,\cdots$. For simplicity, let

$$U_j=\frac{X_{ijl}-\overline{X}_{i.l}}{\sigma_{0l}},\ V_j=\frac{X_{ijl'}-\overline{X}_{i.l'}}{\sigma_{0l'}},\quad \text{and } W_{j'}=\frac{X_{ij'l'}-\overline{X}_{i.l'}}{\sigma_{0l'}}.$$

Then, they can be expressed that

$$(U_j,V_j)\sim N_2(0,0,(n-1)/n,(n-1)/n,\rho)$$

and

$$(U_j,W_j)\sim N_2(0,0,(n-1)/n,(n-1)/n,-\rho/(n-1)).$$

Thus

$$\begin{aligned} Cov(Z_{il},Z_{il'}) &= Cov\left[Z_{il}+(n-1),Z_{il'}+(n-1)\right] \\ &= Cov\left[\sum_{j=1}^{n}U_j{}^2,\sum_{j=1}^{n}V_j{}^2\right] \\ &= n\ \ Cov\left[U_j{}^2,V_j{}^2\right]+n(n-1)Cov\left[U_j,W_j\right]. \end{aligned}$$

By using the moment generating fuction of the bivariate normal distribution, it can be shown that

$$Cov[U_j{}^2,V_j{}^2]=2[Cov(U_j,V_j)]^2,$$

and

$$Cov[U_j{}^2,W_j{}^2]=2[Cov(U_j,W_j)]^2.$$

Hence

$$Cov[Z_{il},Z_{il'}]=2\frac{(n-1)^2}{n}\rho^2+2\frac{(n-1)}{n}\rho^2=2(n-1)\rho^2.$$

Therefore $E(\underline{Z}_i)=\underline{0}$ and $\Sigma_{\underline{Z}}=Cov(\underline{Z}_i)=2(n-1)R^2$.

The following Theorem 2 gives the asymptotic distribution of $\underline{Y_k}$ given by (2.5) when the process is in control.

Theorem 2. Let p-component vectors $\underline{X_1}$, $\underline{X_2}$, $\cdots$ be independently identically distributed according to $N_p(\underline{\mu},\Sigma)$. Then $\left\{\Sigma_{\underline{Y_k}}{}^{-1/2}\underline{Y_k},\ k\geq 1\right\}$ converges in distribution to a multivariate normal distribution with mean vector $\underline{0}$ and variance-covariance matrix I_p as $k \to \infty, \lambda \to 0$ and $k\lambda \to 1$.

Proof. Recall that $\underline{Y}_k$ is

$$\underline{Y_k} = \sum_{i=1}^{k} \lambda(1-\lambda)^{k-i} \underline{Z_i} \,.$$

For $k \geq 1$, let

$$A_k = \frac{1}{k} \sum_{i=1}^{k} Cov\left(\lambda(1-\lambda)^{k-i} \underline{Z_i}\right),$$

B_k is the symmetric, positive definite matrix satisfying ${B_k}^2 = {A_k}^{-1}$,

γ_k = the smallest eigenvalue of A_k.

By the corollaries 18.2 and 18.3 of Bhattacharya and Rao (1975), if

$$\Theta_k(\lambda) = k^{-3/2} \sum_{i=1}^{k} E\|B_k \lambda(1-\lambda)^{k-i} \underline{Z}_i\|^3 \longrightarrow 0$$

as $k \to \infty, \quad \lambda \to 0, \quad$ and $\quad k\lambda \to 1,$

then

$$\frac{1}{\sqrt{k}} B_k \sum_{i=1}^{k} \lambda(1-\lambda)^{k-i} \underline{Z_i} \xrightarrow{\text{d}} N_p(\underline{0}, I), \text{ as } k \to \infty, \ \lambda \to 0, \ k\lambda \to 1.$$

The inequality given (17.63) of Bhattacharya and Rao (1975) is

$$\|B_k \underline{Z}_i\| \leq {\gamma_k}^{-1/2} \|\underline{Z}_i\|, \qquad 1 \leq i \leq k$$

and this gives

$$\|B_k \underline{Z}_i\|^3 \leq {\gamma_k}^{-3/2} \|\underline{Z}_i\|^3, \qquad 1 \leq i \leq k.$$

Thus

$$\begin{aligned} \Theta_k(\lambda) &= k^{-3/2} \sum_{i=1}^{k} E\|B_k \lambda(1-\lambda)^{k-i} \underline{Z}_i\|^3 \\ &= k^{-3/2} \lambda^3 \sum_{i=1}^{k} (1-\lambda)^{3(k-i)} E\|B_k \underline{Z}_i\|^3 \\ &\leq k^{-3/2} \lambda^3 \sum_{i=1}^{k} (1-\lambda)^{3(k-i)} {\gamma_k}^{-3/2} E\|\underline{Z}_i\|^3. \qquad (2.6) \end{aligned}$$

Now

$$A_k = \frac{1}{k}\Sigma_{\underline{Y_k}} = \frac{1}{k}\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2k}\right]\Sigma_{\underline{Z}}\,.$$

Let γ be the smallest eigenvalue of $\Sigma_{\underline{Z}}$. Then the smallest eigenvalue of A_k can be expressed as

$$\gamma_k = \left\{\frac{1}{k}\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2k}\right]\right\}\gamma\,.$$

Thus, the right hand side of the inequality (2.6) is less than or equal to

$$\left[\left\{\frac{1}{k}\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2k}\right]\right\}\gamma\right]^{-3/2}\lambda^3\sum_{i=1}^{k}(1-\lambda)^{3(k-i)}E\|\underline{Z}_i\|^3. \qquad (2.7)$$

By using the inequality given by Chung (1974, p.48), it is easy to show that

$$E\|\underline{Z}_i\|^3 = E\left(\sum_{l=1}^{p} Z_i l^2\right)^{3/2} \le \sqrt{p}\sum_{l=1}^{p} E|Z_{il}|^3 = p^{3/2}E|Z_{il}|^3,\ l=1,2,\cdots,p.$$

Let $m_3 = E|Z_{il}|^3 < \infty,\quad l = 1,2,\cdots,p.$ Thus, the quantity (2.7) is less than or equal to

$$\left(\frac{p}{\gamma}\right)^{3/2} m_3\frac{\sqrt{\lambda}(1-(1-\lambda)^{3k})}{(\lambda^2-3\lambda+3)}\left[\frac{2-\lambda}{1-(1-\lambda)^{2k}}\right]^{3/2}\longrightarrow 0,$$

as $k\to\infty,\quad \lambda\to 0,\quad$ and $\quad k\lambda\to 1.$

Therefore

$$\frac{1}{\sqrt{k}}B_k\sum_{i=1}^{k}\lambda(1-\lambda)^{k-i}\underline{Z_i} = \Sigma_{\underline{Y_k}}{}^{-1/2}\underline{Y_k},\xrightarrow{\ \mathrm{d}\ } N_p(\underline{0},I),$$

as $k\to\infty,\quad \lambda\to 0,\quad$ and $\quad k\lambda\to 1.$

Corollary. $\{T_k{}^2, k\ge 1\}$ converges in distribution to a chi-squared distribution with p degrees of freedom as $k\to\infty, \lambda\to 0,$ and $k\lambda\to 1$.

Proof. Recall that the control statistic ${T_k}^2$ is

$$ {T_k}^2 = \underline{Y_k}'\Sigma_{\underline{Y_k}}{}^{-1}\underline{Y_k} , $$

which can be expressed as

$$ {T_k}^2 = (\Sigma_{\underline{Y_k}}{}^{-1/2}\underline{Y_k})\prime(\Sigma_{\underline{Y_k}}{}^{-1/2}\underline{Y_k}). $$

By Theorem 2 and the corollary of Serfling (1980, p.25),

$$ {T_k}^2 = \underline{Y_k}'\Sigma_{\underline{Y_k}}{}^{-1}\underline{Y_k} \xrightarrow{d} \chi^2(p), \quad \text{as } k \to \infty, \lambda \to 0, k\lambda \to 1. $$

The multivariate EWMA chart signals that the process is out-of-control whenever ${T_k}^2 \geq h$. For the VSI multivariate EWMA chart, suppose that

d_1 is used when ${T_k}^2 \in (g, h]$,

d_2 is used when ${Y_k}^2 \in (0, g]$. (8)

The ATS performance of the multivariate EWMA chart based on accumulate-combine approach can not be modeled as a simple stationary Markov chain as described in Brook and Evans (1972). A simulation to obtain the ATS values and parameters h and g was used.

3. Numerical Results

The following control procedures will be compared on the basis of their ATS performances.

1. FSI multivariate EWMA chart.
2. VSI multivariate EWMA chart.

The performances of the charts for monitoring a variance-covariance matrix depend on the value Σ. It is not possible to investigate all of the different ways in which Σ could be changed. Thus the following types of shifts are considered:

(V1) all variances and covariances are changed by a constant factor, i.e., $\Sigma_1 = c\Sigma_0$.

(V2) one variance is increased to $c1\sigma_{0ii}$ and the other variances are remained on target.

(V3) approximately half of the variances and covariances are changed by a constant and there are no shifts in the rest.

When comparing FSI and VSI charts, some kinds of standard for comparison is necessary. The charts are matched for ANSS and ATS when the process is in control. This enables the performance to be evaluated when the process has shifted away from its target value. For convenience the unit of time was chosen as the sampling interval of the FSI chart so that $d = 1$. When $d = 1$, the ANSS and ATS of the FSI chart have the same value. By using the relationships between FSI and VSI sampling intervals, two sampling intervals d_1 and d_2 for the VSI charts can be chosen so that two charts have the same ATS when the process is in control. In our computation, the ANSS in control was fixed to be 200 and all of the VSI charts used d_1=0.1 and d_2 =1.9.

The sample size used for each sample observations was five. It is assumed that the correlation coefficient ρ is the same for all variables. Table 1 gives the values of h and g for p =2-5 and λ when the ATS at $\Sigma = \Sigma_0$ is approximately 200. ATS values and parameters h and g were calculated by using 10,000 simulations. For p=2 and three different correlation coefficients ρ =0.0, 0.5, 0.8, Tables 2-4 give FSI and VSI ATS values. As shown in Tables 2-4, smaller values of λ are more effective in detecting all shifts in Σ for p=2. For p=5, ρ = 0.5, and λ = 0.05, Table 5 gives FSI and VSI ATS values. The results in Tables 2-5 show that VSI multivariate EWMA chart based on accumulate-combine approach is better than FSI multivariate EWMA chart based on combine-accumulate approach.

Table 1. Values of h and g for Various Values of λ and p when the ATS at $\Sigma = \Sigma_0$ is approximately 200

p	ρ	$\lambda = 0.3$	$\lambda = 0.1$	$\lambda = 0.05$
2	0.0	14.0154 1.2730	9.4105 1.2790	7.7675 1.2390
2	0.5	14.3502 1.2560	9.5307 1.2710	7.8020 1.2210
2	0.8	14.3502 1.2560	9.5307 1.2710	7.8020 1.2210
3	0.5			10.0917 2.1340
4	0.5			12.1742 3.0670
5	0.5			14.0632 3.9810

The top number in each cell is h

The bottom number in each cell is g

Table 2. ATS Values for Matched Multuvariate FSI and VSI EWMA Charts for the Variance-Covariance Matrix ($p = 2, \rho = 0.0$)

	$\lambda = 0.05$		$\lambda = 0.10$		$\lambda = 0.30$	
shift	FSI	VSI	FSI	VSI	FSI	VSI
$c = 1.00$	199.5	200.0	199.6	199.9	199.5	200.0
$c = 1.21$	21.7	16.5	24.4	19.2	34.4	29.4
$c = 1.69$	4.1	2.9	4.6	3.1	6.0	3.6
$c = 2.56$	2.1	1.6	2.2	1.7	2.7	1.8
$c1 = 1.21$	38.4	31.5	42.9	36.7	60.7	55.7
$c1 = 1.69$	7.7	5.4	8.5	5.9	11.7	8.0
$c1 = 2.56$	2.9	2.2	3.2	2.3	3.9	2.5

Table 3. ATS Values for Matched Multivariate FSI and VSI EWMA Charts for the Variance-Covariance Matrix ($p = 2, \rho = 0.5$)

	$\lambda = 0.05$		$\lambda = 0.10$		$\lambda = 0.30$	
shift	FSI	VSI	FSI	VSI	FSI	VSI
$c = 1.00$	200.6	199.3	200.5	200.0	200.2	199.8
$c = 1.21$	24.9	19.4	27.8	22.5	38.6	34.5
$c = 1.69$	4.7	3.3	5.3	3.5	6.9	4.3
$c = 2.56$	1.9	1.5	2.1	1.6	2.4	1.6
$c1 = 1.21$	37.4	29.8	42.5	35.7	61.7	55.7
$c1 = 1.69$	7.4	5.1	8.2	5.5	11.4	7.6
$c1 = 2.56$	2.8	2.1	3.1	2.2	3.8	2.4

Table 4. ATS Values for Matched Multivariate FSI and VSI EWMA Charts for the Variance-Covariance Matrix ($p = 2, \rho = 0.8$)

	$\lambda = 0.05$		$\lambda = 0.10$		$\lambda = 0.30$	
shift	FSI	VSI	FSI	VSI	FSI	VSI
$c = 1.00$	200.1	198.7	201.7	201.2	200.8	200.7
$c = 1.21$	28.5	23.2	31.5	26.4	42.0	36.9
$c = 1.69$	5.4	3.8	6.0	4.1	7.9	5.1
$c = 2.56$	2.1	1.7	2.3	1.7	2.7	1.8
$c1 = 1.21$	27.9	20.2	33.0	25.0	52.0	43.9
$c1 = 1.69$	5.4	3.6	6.1	3.8	8.3	4.8
$c1 = 2.56$	2.2	1.7	2.4	1.7	2.8	1.8

Table 5. ATS Values for Matched Multivariate FSI and VSI EWMA Charts for the Variance-Covariance Matrix ($p = 5, \rho = 0.5, \lambda = 0.05$)

shift	FSI	VSI	shift	FSI	VSI
$c = 1.00$	200.4	199.4	c1-3 = 1.21	22.7	15.6
$c = 1.21$	18.6	13.0	c1-3 = 1.44	7.5	4.8
$c = 1.69$	3.2	2.2	c1-3 = 1.69	4.1	2.7
$c = 2.56$	1.4	1.2	c1-3 = 1.96	2.7	1.9
c1 = 1.21	46.3	35.7	c1-3 = 2.25	2.0	1.5
c1 = 1.69	8.5	5.7	c1-3 = 2.56	1.7	1.3
c1 = 2.56	3.2	2.4			

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