Youngnam Statistical Letters
June 1993, Vol. 4, 13-30

Statistical Inferences in the Weibull Regression Model based on Censored Data [1]

Kil-Ho Cho[2]

ABSTACT

We propose the ordered least squares estimators(OLSE's) of the parameters and the $p-$th quantiles for the two-parameter Weibull regression model under the Type II censoring. The Monte Carlo simulations are performed to compare the proposed estimators with the maximum likelihood estimators(MLE's), and it is shown that the proposed estimators are slightly better than MLE's as the censoring rate goes up.

1. Introduction

The two-parameter Weibull regression model has been widely used in the filed of the reliability and life testing. The probability density function(p.d.f.) of the Weibull regression model for lifetime T, given the vector $\underline{x}$ of covariates,is given by

$$f(t|\underline{x}) = \frac{\delta}{\alpha(\underline{x})}\left[\frac{t}{\alpha(\underline{x})}\right]^{\delta-1} \exp\left[-(\frac{t}{\alpha(\underline{x})})^{\delta}\right], \quad t \geq 0 \tag{1.1}$$

[1] This paper was supported by Non Directed Research Fund, Korea Research Foundation, 1991.

[2] Department of Statistics, Kyungpook National University, Taegu, 702-701, Korea.

where $\alpha(\underline{x})$ is the scale parameter that depends on $\underline{x}$ and δ is the shape parameter independent of $\underline{x}$. The fact that δ does not depend on $\underline{x}$ implies proportional hazards for lifetimes and constant variance for log lifetimes of individuals. This assumption is reasonable in many situations.(See Pike(1966) , Peto and Lee(1973), and Nelson(1972).)

We shall work with log lifetime ;

The p.d.f. of $Y = \log T$, given $\underline{x}$, is

$$f(y|\underline{x}) = \frac{1}{\sigma}\exp\left[\frac{y-\mu(\underline{x})}{\sigma} - \exp\left(\frac{y-\mu(\underline{x})}{\sigma}\right)\right], \quad -\infty < y < \infty \tag{1.2}$$

where $\mu(\underline{x}) = \log\alpha(\underline{x})$ and $\sigma = \frac{1}{\delta}$. The most frequently used model is the linear one with

$$\mu(\underline{x}) = \underline{x}\underline{\beta}$$

where, $\underline{x} = (x_0, x_1, \ldots, x_p)$ and $\underline{\beta} = (\beta_0, \beta_1, \ldots, \beta_p)'$.

Nelson (1972) and Nelson and Hahn (1972) discussed estimation for parameters of the Weibull regression model with a single regressor variable. Prentice and Shillington (1975) presented least squares methods for uncensored Weibull data. Mann (1978) studied confidence interval estimation in the model with a single regression model. Williams (1978) considered a least squared type of procedure that has higher asymptotic relative efficiency but that likewise cannot be used with censored data.

We, in this paper, propose the ordered least squares estimators of the parameters and the p−th quantiles for the two-parameter Weibull regression model under the Type II censoring. Also, through the Monte Carlo simulations, we compare the proposed estimators with other estimators and investigate small sample properties of them. In Section 2, the ordered least squares estimators of the parameters and the p−th quantile for the model (1.2) under the Type II censoring are derived. In Section 3, the biases and estimated mean square errors (MSE's) of the proposed estiomators and the maximum likelihood estimators are compared by using Monte Carlo simulations. Finally, the data of cancer for male mice is used to illustrate for the behaviors of the proposed estimators in Section 4.

2. Estimation of parameters and quantiles under the Type II censoring

Sometimes in life test experiments to investigate the effect of a small number of factors on lifetime, there may be several observations at each $\underline{x}$. Suppose that observations are taken on n_i individual at $\underline{x}_i = (x_{i0}, x_{i1}, \ldots, x_{ip})$, $i = 1, 2, \ldots, m$; we will allow the sample at $\underline{x}_i$ to be Type II censored, so that just the first r_i ordered lifetimes $t_{i(1)} \leq t_{i(2)} \leq \cdots \leq t_{i(r_i)}$ out of the total of n_i are observed.

Let

$$y_{i(j)} = \log t_{i(j)}$$

Then the log likelihood function for the model (1.2) is

$$\begin{aligned}\log L(\underline{\beta}, \sigma) &= \sum_{i=1}^{m} \log\left[\frac{n_i!}{(n_i - r_i)!}\right] - r \log \sigma + \sum_{i=1}^{m} \sum_{j=1}^{r_i} \left(\frac{y_{i(j)} - \underline{x}_i \underline{\beta}}{\sigma}\right) \\ &- \sum_{i=1}^{m} \left[\sum_{j=1}^{r_i} \exp\left(\frac{y_{i(j)} - \underline{x}_i \underline{\beta}}{\sigma}\right) + (n_i - r_i) \exp\left(\frac{y_{i(r_i)} - \underline{x}_i \underline{\beta}}{\sigma}\right)\right] \\ &\equiv \sum_{i=1}^{m} \log\left[\frac{n_i!}{(n_i - r_i)!}\right] - r \log \sigma + \sum_{i=1}^{m} \sum_{j=1}^{r_i} \left(\frac{y_{i(j)} - \underline{x}_i \underline{\beta}}{\sigma}\right) \\ &- \sum_{i=1}^{m} \sum_{j=1}^{r_i}{}^{*} \exp\left(\frac{y_{i(j)} - \underline{x}_i \underline{\beta}}{\sigma}\right) \end{aligned} \tag{2.1}$$

where

$$r \equiv \sum_{i=1}^{m} \sum_{j=1}^{r_i} = \sum_{i=1}^{m} r_i$$

$$\sum_{j=1}^{r_i}{}^{*} a_{i(j)} \equiv \sum_{j=1}^{r_i} a_{i(j)} + (n_i - r_i) a_{i(r_i)}$$

If we let

$$Z_{i(j)} \equiv [y_{i(j)} - \underline{x}_i \underline{\beta}]/\sigma,$$

the first derivatives of $\log L(\underline{\beta}, \sigma)$ are

$$\frac{\partial \log L(\underline{\beta}, \sigma)}{\partial \beta_l} = -\frac{1}{\sigma} \sum_{i=1}^{m} r_i x_{il} + \frac{1}{\sigma} \sum_{i=1}^{m} x_{il} \sum_{j=1}^{r_i}{}^{*} e^{z_{i(j)}} \quad , l = 0, 1, \ldots, p \tag{2.2}$$

$$\frac{\partial \log L(\underline{\beta},\sigma)}{\partial \sigma_l} = -\frac{r}{\sigma} - \frac{1}{\sigma}\sum_{i=1}^{m}\sum_{j=1}^{r_i} Z_{i(j)} + \frac{1}{\sigma}\sum_{i=1}^{m}\sum_{j=1}^{r_i}{}^{*}\left(z_{i(j)}e^{z_{i(j)}}\right) \tag{2.3}$$

The maximum likelihood equations $\partial \log L(\underline{\beta},\sigma)/\partial\beta_l = 0 \quad (l = 0, 1, \ldots, p)$ and $\partial \log L(\underline{\beta},\sigma)/\partial\sigma = 0$ can be solved by iterative procedure(Lawless(1982)). We will denote the solutions of the equations by $\underline{\hat{\beta}} = (\hat{\beta}_0, \hat{\beta}_1, \cdots, \hat{\beta}_p)'$ and $\hat{\sigma}$.

If there are enough observations at each $\underline{x}$, the ordered least squares procedure can be used. Now, we find the ordered least square estimators of parameters.

Lemma 1. Suppose that $y_1, \cdots, y_n$ is a random sample of size n from a distribution of the model (1.2). Let

$$z_i = \frac{y_i - \underline{x}\underline{\beta}}{\sigma}, \quad i = 1, \ldots, n.$$

Then Z_i has a standard extrem value distribution with p.d.f.

$$f(z) = \exp(z - e^z), \quad -\infty < z < \infty \tag{2.4}$$

Lemma 2. (Lieblein(1953)) Suppose that $z_{(1)} \le \cdots \le z_{(r)}$ are the smallest r observations in a sample of size n from a distribution of the model (2.4). Put

$$\underline{Z} = (z_{(1)}, z_{(2)}, \cdots, z_{(r)})'$$

Then the mean vector of $\underline{Z}$ is

$$E(\underline{Z}) \equiv \underline{\alpha} = (\alpha_1, \cdots, \alpha_r)' \tag{2.5}$$

with

$$\alpha_i \equiv E[Z_{(i)}] = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}g_1(i+j)$$

where

$$g_1(c) = \frac{1}{c}(\gamma + \log c)$$

$$\gamma = -\Gamma'(1) \text{ is Euler's constant }, \quad 0.5772156649\cdots.$$

The variance-covariance matrix of $\underline{Z}$ is

$$V(\underline{Z}) \equiv V = (v_{ij}), \quad i,j = 1,2,\dots,r \tag{2.6}$$

with

$$\begin{aligned} v_{ii} &= E[Z_{(i)}{}^2] - [E[Z_{(i)}]]^2 \\ &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}g_2(i+j) - \alpha_i{}^2 \qquad i = 1,\dots,r \\ v_{ij} &= E[Z_{(i)}Z_{(j)}] - E[Z_{(i)}]E[Z_{(j)}] \\ &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\sum_{r=0}^{j-i-1}\sum_{s=0}^{n-j}(-1)^{r+s}\binom{j-i-1}{r}\binom{n-j}{s} \\ &\quad \times \phi(i+r, j-i-r+s) - \alpha_i\alpha_j \qquad i<j, \quad i,j = 1,\dots,r \end{aligned}$$

where

$$g_2(c) = \frac{1}{c}\left[\frac{\pi^2}{6} + (\gamma + \log c)^2\right]$$

$$\phi(t,u) = \begin{cases} \frac{1}{2u^2}(\gamma + \log u)^2, & \text{if } t = u \\ \frac{1}{2tu}\left[(u-t)g_2(t+u) + t^2[g_1(t)]^2 + 2L(1+\frac{t}{u}) - \left(\log\frac{t}{u}\right)^2 - \frac{\pi^2}{6}\right], & \text{if } t < u \\ \frac{1}{2tu}\left[(u-t)g_2(t+u) + t^2[g_1(t)]^2 - 2L(1+\frac{t}{u}) + \frac{\pi^2}{6}\right], & \text{if } t > u \end{cases}$$

$$L(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}x^n$$

Note that the mean vector and variance-covariance matrix of $\underline{Z}_i$ at $\underline{x}_i$, say $\underline{\alpha}_i$ and V_i , are independent of $\underline{\mu}_i = \mu(\underline{x}_i)$ and σ.

Lemma 3. If we let

$$\underline{y}_i = (y_{i(1)}, \cdots, y_{i(r_i)})', \quad i = 1,\dots,m,$$

then the mean vector and variance-covariance matrix of $\underline{y}_i$ are given by ;

$$E(\underline{y}_i) = \mu_i\underline{1} + \sigma\underline{\alpha}_i \equiv (\underline{1}, \underline{\alpha}_i)\binom{\mu_i}{\sigma} \tag{2.7}$$

$$V(\underline{y}_i) = \sigma^2 V(\underline{Z}_i) \equiv \sigma^2 V_i \tag{2.8}$$

where $\underline{1}$ is $1 \times r_i$ vector of units.

The ordered least squares estimation is the least squares method of using the order statistics.

Theorem 1. The ordered least squares estimators of μ_i and σ for the model (1.2) at $\underline{x}_i$ under Type II censoring are given by

$$\tilde{\mu}_i = -\underline{\alpha}_i{}'V_i{}^{-1}(\underline{1}\ \underline{\alpha}_i{}' - \underline{\alpha}_i\ \underline{1}')V_i{}^{-1}\underline{y}_i/\Delta \tag{2.9}$$

$$\tilde{\underline{\sigma}}_i = \underline{1}'V_i{}'(\underline{1}\ \underline{\alpha}_i{}' - \underline{\alpha}_i\ \underline{1}')V_i{}^{-1}\underline{y}_i/\Delta \tag{2.10}$$

where

$$\Delta = (\underline{1}'V_i{}^{-1}\underline{1})(\underline{\alpha}_i{}'V_i{}^{-1}\underline{\alpha}_i) - (\underline{1}'V_i{}^{-1}\underline{\alpha}_i)^2$$

proof. From Lemma 3, the least squares estimators of μ_i and σ at $\underline{x}_i$ are

$$\begin{pmatrix}\tilde{\mu}_i\\ \tilde{\sigma}_i\end{pmatrix} = \left[(\underline{1}\ \underline{\alpha}_i)'V_i{}^{-1}(\underline{1}\ \underline{\alpha}_i)\right]^{-1}(\underline{1}\ \underline{\alpha}_i)'V_i{}^{-1}\underline{y}_i$$

$$= \frac{1}{\Delta}\begin{bmatrix}-\underline{\alpha}_i{}'V_i{}^{-1}(\underline{1}\ \underline{\alpha}_i{}' - \underline{\alpha}_i\ \underline{1}')\\ \underline{1}'V_i{}^{-1}(\underline{1}\ \underline{\alpha}_i{}' - \underline{\alpha}_i\ \underline{1}')\end{bmatrix}V_i{}^{-1}\underline{y}_i$$

where

$$\Delta = (\underline{1}'V_i{}^{-1}\underline{1})(\underline{\alpha}_i{}'V_i{}^{-1}\underline{\alpha}_i) - 3(\underline{1}'V_i{}^{-1}\underline{\alpha}_i)^2$$

Lemma 4.

(1) The ordered least squares estimators $\tilde{\mu}_i$ and $\tilde{\sigma}_i$ are unbiased for μ_i and σ, respectively.

(2) The variances of $\tilde{\mu}_i$ and $\tilde{\sigma}_i$ are

$$Var(\tilde{\mu}_i) = \sigma^2\underline{\alpha}_i{}'V_i{}^{-1}\underline{\alpha}_i/\Delta \tag{2.11}$$

and

$$Var(\tilde{\sigma}_i) = \sigma^2\underline{1}'V_i{}^{-1}\underline{1}/\Delta \tag{2.12}$$

The covarince of $\tilde{\mu}_i$ and $\tilde{\sigma}_i$ is

$$Cov(\tilde{\mu}_i, \tilde{\sigma}_i) = -\sigma^2\underline{1}'V_i{}^{-1}\underline{\alpha}_i/\Delta \tag{2.13}$$

Now, we find the ordered least squares estimator of $\underline{\beta}$.

Theorem 2. The ordered least squares estimator of $\underline{\beta}$ is given by

$$\underline{\tilde{\beta}} = (X'WX)^{-1}X'W\underline{\tilde{\mu}} \tag{2.14}$$

where $X = [\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_n]'$ with $\underline{x}_i = (x_{i0}, x_{i1}, \cdots, x_{ip})'$, and $W = \text{diag}(w_1, w_2, \cdots, w_m)$ with $w_i = \dfrac{\sigma^2}{Var(\tilde{\mu}_i)} = \dfrac{\Delta}{\underline{\alpha}_i{}'V_i^{-1}\underline{\alpha}_i}$, $\underline{\tilde{\mu}} = (\tilde{\mu}_1, \cdots, \tilde{\mu}_m)'$.

proof. The estimator is obtained by minimizing $\sum_{i=1}^{m} w_i(\tilde{\mu}_i - \underline{x}_i\underline{\beta})^2$.

This estimator is unbiased for $\underline{\beta}$ and has the variance-covariance matrix

$$V(\underline{\tilde{\beta}}) = \sigma^2(X'WX)^{-1}. \tag{2.15}$$

Theorem 3. The minimum variance unbiased estimator of σ that is a linear combination of $\tilde{\sigma}_1, \cdots, \tilde{\sigma}_m$ is

$$\tilde{\sigma} = \sum_{i=1}^{m} b_i^{-1}\tilde{\sigma}_i / (\sum_{i=1}^{m} b_i^{-1}) \tag{2.16}$$

where

$$b_i^{-1} = \frac{\sigma^2}{Var(\tilde{\sigma}_i)} = \frac{\Delta}{\underline{1}'V_i^{-1}\underline{1}}$$

proof. Assume that $\tilde{\sigma} = \sum_{i=1}^{m} k_i\tilde{\sigma}_i$. We must minimize the variance of $\tilde{\sigma}$, say $Var(\tilde{\sigma})$, subject to $\sum_{i=1}^{m} k_i = 1$. Using Lagrange multiplier method, we can get the equation (2.16).

Next, we consider the estimator of the $p-$th quantile of the model (1.2).

The $p-$th quantile of the model (1.2) is given by

$$y_p(\underline{x}) = \underline{x}\underline{\beta} + \sigma \log[-\log(1-p)], \quad 0 < p < 1. \tag{2.17}$$

Using the estimators of parameters, we can obtain the following estimators of quantiles.

From the invariance property of the MLE, the MLE of the p-th quantile of the model (1.2) is given by

$$\hat{y}_p(\underline{x}) = \underline{x}\hat{\underline{\beta}} + \hat{\sigma}\log[-\log(1-p)], \quad 0 < p < 1. \tag{2.18}$$

Also,using the ordered least squares estimators of $\underline{\beta}$ and σ, we get an estimator of $y_p(\underline{x})$ given by

$$\tilde{y}_p(\underline{x}) = \underline{x}\tilde{\underline{\beta}} + \tilde{\sigma}\log[-\log(1-p)], \quad 0 < p < 1. \tag{2.19}$$

3. Simulations

In this section, using the Monte Carlo simulations, we compare the performances of the proposed estimators with MLE's in terms of bias and estimated mean square error(MSE), and investigate the small sample properties of them.

The simulations are performed for some combinatons of three lifetime distributions $(\sigma = 0.5, 1, 2)$, censoring rate $(10\%, 30\%, 50\%)$, and sample size at each $\underline{x}_i$ $(n_i = 10, 20, 30)$ on 386PC. Each simulation consists of 500 replications by using the IMSL subroutine 'GGUBS'.

Simulation results are shown in Table 3.1 - 3.7, which contain the bias, estimated MSE, and standard error of MSE(SDMSE).

In respect to Table 3.1 - 3.7, we note:

(1) It seems that the MLE's of parameters and quantiles are underestimated.

(2) Estimators of two types have unsensitive properties for the given types of distributions.

(3) The MSE's of all estimators get smaller as the size of sample, n_i, goes up for each fixed censoring rate, and also get smaller as the censoring rate goes down for each fixed sample size.

(4) For parameters and quantiles, the MSE's of OLSE's are smaller than those of MLE's as the censoring rate goes up. From the above facts, we can conclude that OLSE's are better than MLE's as the censoring rate goes up.

4. Example

We look at some results of applying the proposed estimators and MLE's. As an example, we consider the data of days until occurrence of cancer for male mice in Data set IV of Kalbfleish and Prentice(1980). In this trials, the first group of mice (Table 4.1) lived in a conventional laboratory environment while the second group (Table 4.2) was in a germ-free environment.

We assume that the data are censored about 50% at each $\underline{x}_i$. A regression vector $\underline{x} = (x_0, x_1, x_2, x_3)$ with each individual is as follows;

$x_0 = 1$

$x_1 = 1$ if turmor type is thymic lymphoma, 0 otherwise.

$x_2 = 1$ if turmor type is reticulum cell sarcoma, 0 otherwise.

$x_3 = 1$ if treatment is control group, 0 if it is germ-free group.

Estimates of two types of methods are given as follows;

TYPE	Estimates of β_0	β_1	β_2	β_3	σ
OLSE	0.360	37.172	142.140	-37.740	38.667
MLE	0.337	35.428	136.435	-35.258	32.114

Table 3.1 Biases and MSE's for estimators of β and σ

(True value ; $\underline{\beta} = (1,2)$, $\sigma = 0.5$)

Ni	Censoring ratio	TYPE	Estimators of β_0 Bias	MSE	SDMSE	Estimators of β_1 Bias	MSE	SDMSE	Estimators of σ Bias	MSE	SDMSE
	10 %	OLSE	.000	.407	.002	.000	1.245	.006	.000	.079	.000
		MLE	-.375	.403	.002	-.758	1.257	.007	-.027	.081	.000
10	30 %	OLSE	.000	.748	.004	.001	1.978	.020	.000	.491	.002
		MLE	-.402	.763	.005	-.817	1.989	.023	-.064	.504	.002
	50 %	OLSE	.000	.834	.017	.002	3.301	.053	.000	.771	.004
		MLE	-.878	.892	.018	-.912	4.323	.057	-.209	.865	.005
	10 %	OLSE	.000	.228	.012	.000	.716	.002	.000	.063	.000
		MLE	-.338	.215	.010	-.693	.721	.004	-.020	.061	.000
20	30 %	OLSE	.000	.352	.001	.000	1.382	.003	.000	.317	.001
		MLE	-.401	.373	.002	-.749	1.397	.004	-.039	.347	.001
	50 %	OLSE	.000	.761	.001	.000	2.941	.004	.000	.664	.002
		MLE	-.856	.876	.002	-.866	3.125	.006	-.182	.743	.003
	10 %	OLSE	.000	.216	.000	.000	.602	.000	.000	.028	.000
		MLE	-.313	.218	.000	-.431	.613	.001	-.014	.023	.000
30	30 %	OLSE	.000	.612	.000	.000	1.026	.001	.000	.235	.001
		MLE	-.384	.643	.001	-.717	1.118	.002	-.018	.258	.001
	50 %	OLSE	.000	.637	.002	.000	2.427	.003	.000	.643	.002
		MLE	-.892	.713	.003	-.809	2.842	.004	-.143	.718	.003

Table 3.2 Biases and MSE's for estimators of quantiles at $x = (1,0)$ when censoring ratio is 10 %

(True value ; $\underline{\beta} = (1,2)$, $\sigma = 0.5$)

P	TYPE	Quantile	$N_i = 10$ Bias	MSE	SDMSE	$N_i = 20$ Bias	MSE	SDMSE	$N_i = 30$ Bias	MSE	SDMSE
.05	OLSE	-.485	.000	.072	.002	.000	.071	.001	.000	.067	.001
	MLE	-.485	-.257	.074	.002	-.240	.071	.001	-.231	.065	.001
.1	OLSE	-.125	.000	.046	.001	.000	.043	.000	.000	.041	.000
	MLE	-.125	-.004	.044	.001	-.002	.043	.001	-.002	.042	.000
.3	OLSE	.484	.000	.057	.005	.000	.051	.003	.000	.049	.002
	MLE	.484	-.113	.053	.005	-.111	.052	.003	-.103	.041	.002
.5	OLSE	.816	.000	.068	.003	.000	.061	.001	.000	.057	.001
	MLE	.816	-.064	.070	.003	-.063	.063	.002	-.061	.052	.002
.7	OLSE	1.092	.000	.113	.002	.000	.097	.000	.000	.086	.000
	MLE	1.092	-.023	.124	.003	-.019	.103	.001	-.011	.089	.000
.9	OLSE	1.417	.000	.204	.003	.000	.201	.001	.000	.176	.000
	MLE	1.417	-.024	.215	.004	-.022	.203	.001	-.021	.169	.001
.95	OLSE	1.548	.000	.207	.004	.000	.202	.000	.000	.200	.001
	MLE	1.548	-.043	.213	.005	-.033	.211	.003	-.029	.198	.001

Table 3.3 Biases and MSE's for estimators of quantiles at $x = (1,1)$ when censoring ratio is 10 %

(True value ; $\underline{\beta} = (1,2)$, $\sigma = 0.5$)

P	TYPE	Quantile	$N_i = 10$			$N_i = 20$			$N_i = 30$		
			Bias	MSE	SDMSE	Bias	MSE	SDMSE	Bias	MSE	SDMSE
.05	OLSE	1.514	.000	.649	.004	.000	.196	.001	.000	.083	.000
	MLE	1.514	-.320	.657	.005	-.157	.209	.002	-.146	.097	.001
.1	OLSE	1.874	.000	.747	.003	.000	.283	.001	.000	.261	.001
	MLE	1.874	-.515	.739	.002	-.366	.275	.001	-.312	.256	.001
.3	OLSE	2.484	.000	1.355	.005	.000	.889	.002	.000	.792	.002
	MLE	2.484	-.995	1.416	.006	-.882	.888	.002	-.767	.789	.002
.5	OLSE	2.816	.000	2.285	.006	.000	1.847	.003	.000	1.674	.003
	MLE	2.816	-1.404	2.417	.009	-1.321	1.824	.003	-1.209	1.702	.003
.7	OLSE	3.092	.001	3.908	.009	.000	3.536	.005	.000	3.312	.003
	MLE	3.092	-1.897	3.925	.010	-1.849	3.716	.007	-1.779	3.086	.002
.9	OLSE	3.417	.001	8.317	.017	.000	8.164	.011	.000	8.024	.009
	MLE	3.417	-1.986	9.417	.021	-1.911	7.924	.010	-1.866	7.325	.008
.95	OLSE	3.548	.001	11.789	.025	.001	11.624	.017	.000	1.427	.010
	MLE	3.548	-2.800	12.541	.027	-2.101	1.114	.016	-1.924	9.428	.009

Table 3.4 Biases and MSE's for estimators of quantiles at $x = (1,0)$ when censoring ratio is 30 %

(True value ; $\underline{\beta} = (1,2)$, $\sigma = 1$)

P	TYPE	Quantile	$N_i = 10$ Bias	MSE	SDMSE	$N_i = 20$ Bias	MSE	SDMSE	$N_i = 30$ Bias	MSE	SDMSE
.05	OLSE	-1.970	.000	1.549	.095	.000	.229	.015	.000	.211	.009
	MLE	-1.970	-.081	1.627	.096	-.022	.251	.017	-.021	.176	.010
.1	OLSE	-1.250	.000	1.191	.072	.000	.204	.012	.000	.178	.007
	MLE	-1.250	-.071	1.207	.073	-.030	.215	.014	-.027	.192	.008
.3	OLSE	-.030	.000	.824	.048	.000	.201	.009	.000	.113	.006
	MLE	-.030	-.054	.832	.051	-.043	.207	.007	-.031	.142	.006
.5	OLSE	.633	.000	.750	.041	.000	.305	.009	.000	.213	.005
	MLE	.633	-.045	.778	.046	-.040	.316	.008	-.037	.232	.004
.7	OLSE	1.185	.000	.757	.041	.000	.379	.012	.000	.301	.007
	MLE	1.185	-.037	.767	.046	-.031	.392	.014	-.030	.303	.007
.9	OLSE	1.834	.000	.843	.048	.000	.493	.016	.000	.387	.0092
	MLE	1.834	-.028	.862	.049	-.020	.507	.017	-.019	.392	.010
.95	OLSE	2.097	.001	.902	.054	.000	1.548	.019	.000	1.471	.011
	MLE	2.097	-.024	.917	.058	-.020	.579	.021	-.019	.493	.013

Table 3.5 Biases and MSE's for estimators of quantiles at $x = (1,1)$ when censoring ratio is 30%

(True value ; $\underline{\beta} = (1,2)$, $\sigma = 1$)

P	TYPE	Quantile	$N_i = 10$ Bias	MSE	SDMSE	$N_i = 20$ Bias	MSE	SDMSE	$N_i = 30$ Bias	MSE	SDMSE
.05	OLSE	.029	.000	.219	.016	.000	.203	.015	.000	.197	.013
	MLE	.029	-.016	.227	.018	-.013	.217	.016	-.011	.209	.015
.1	OLSE	.749	.000	.421	.018	.000	.416	.016	.000	.301	.014
	MLE	.749	-.116	.427	.019	-.114	.419	.018	-.102	.311	.016
.3	OLSE	1.969	.000	.827	.024	.000	.714	.021	.000	.602	.018
	MLE	1.969	-.117	.853	.027	-.115	.752	.026	-.105	.642	.021
.5	OLSE	2.633	.001	1.413	.040	.000	1.401	.031	.000	1.309	.028
	MLE	2.633	-.136	1.446	.042	-.126	1.431	.036	-.118	1.341	.030
.7	OLSE	3.185	.001	1.656	.087	.000	1.517	.067	.000	1.471	.047
	MLE	3.185	-.185	1.689	.089	-.146	1.645	.069	-.139	1.476	.050
.9	OLSE	3.834	.002	2.333	.107	.001	2.200	.096	.000	2.003	.063
	MLE	3.834	-.163	2.592	.112	-.160	2.471	.101	-.147	2.221	.069
.95	OLSE	4.097	.002	3.874	.116	.001	3.317	.101	.001	3.274	.092
	MLE	4.097	-.173	4.102	.119	-.170	3.529	.112	-.158	3.416	.107

Table 3.6 Biases and MSE's for estimators of quantiles at $x = (1, 0)$ when censoring ratio is 50 %

(True value ; $\underline{\beta} = (1, 2)$, $\sigma = 2$)

P	TYPE	Q	$N_i = 10$ Bias	MSE	SDMSE	$N_i = 20$ Bias	MSE	SDMSE	$N_i = 30$ Bias	MSE	SDMSE
.05	OLSE	-4.940	.001	19.317	.098	.000	17.912	.090	.000	6.070	.081
	MLE	-4.940	-.438	22.162	.187	-.421	2.426	.111	-.241	8.425	.097
.1	OLSE	-3.500	.000	9.723	.063	.000	9.019	.063	.000	2.862	.005
	MLE	-3.500	-.310	13.189	.087	-.299	11.242	.079	-.164	3.721	.014
.3	OLSE	-1.061	.000	.941	.019	.000	.861	.011	.000	.190	.000
	MLE	-.061	-.094	1.425	.021	-.091	1.321	.014	-.035	.276	.001
.5	OLSE	.267	.000	.110	.005	.000	.071	.003	.000	.060	.000
	MLE	.267	-.023	.114	.006	-.021	.095	.004	-.020	.072	.000
.7	OLSE	1.371	.000	1.541	.026	.000	1.378	.021	.000	1.186	.002
	MLE	1.371	-.121	1.607	.027	-.115	1.553	.024	-.103	1.392	.002
.9	OLSE	2.668	.000	5.680	.058	.000	5.186	.056	.000	2.822	.006
	MLE	2.668	-.236	6.527	.072	-.225	6.482	.067	-.161	3.927	.012
.95	OLSE	3.194	.001	8.118	.073	.001	7.431	.071	.000	6.858	,008
	MLE	3.194	-.283	1.416	.081	-.270	9.021	.073	-.189	7.428	.018

Q : represent quantile.

Table 3.7 Biases and MSE's for estimators of quantiles at $x = (1,1)$ when censoring ratio is 50 %

(True value ; $\underline{\beta} = (1,2)$, $\sigma = 2$)

P	TYPE	Q	$N_i = 10$ Bias	MSE	SDMSE	$N_i = 20$ Bias	MSE	SDMSE	$N_i = 30$ Bias	MSE	SDMSE
.05	OLSE	-2.940	.001	6.948	.056	.000	6.325	.035	.000	3.274	.007
	MLE	-2.940	-.262	7.242	.057	-.249	6.924	.041	-.173	4.097	.009
.1	OLSE	-1.500	.000	1.861	.025	.000	1.665	.027	.000	1.094	.003
	MLE	-1.500	-.134	2.123	.030	-.126	2.024	.025	-.097	1.367	.002
.3	OLSE	.938	.000	.712	.015	.000	.667	.010	.000	.163	.000
	MLE	.938	-.081	.791	.018	-.061	.742	.013	-.031	.217	.002
.5	OLSE	2.267	.000	4.040	.039	.000	3.795	.025	.000	1.117	.002
	MLE	2.267	-.199	5.278	.041	-.194	4.943	.027	-.102	1.862	.003
.7	OLSE	3.371	.001	8.928	.063	.001	8.348	.049	.000	3.810	.004
	MLE	3.371	-.297	1.417	.079	-.288	9.724	.052	-.191	4.427	.005
.9	OLSE	4.668	.001	17.126	.097	.001	15.968	.095	.000	8.457	.009
	MLE	4.668	-.412	19.927	.199	-.398	17.128	.145	-.229	9.927	.012
.95	OLSE	5.194	.002	21.211	.117	.002	19.762	.119	.001	13.848	.114
	MLE	5.194	-.445	24.197	.219	-.443	22.134	.203	-.257	15.729	.193

Q : represent quantile.

Table 4.1 Data of Control group

TYPE	n_i	r_i	Data
Thymic lymphoma	22	11	159, 189, 191, 198, 200, 207, 220, 235, 245, 250, 256
Reticulum cell sarcoma	38	19	317, 318, 399, 495, 525, 536, 549, 552, 554, 337, 558, 571, 586, 594, 596, 605, 612, 621, 628
Other causes	39	20	40, 42, 51, 62, 163, 179, 206, 222, 228, 252, 249, 282, 324, 333, 341, 366, 385, 407, 420, 431

Table 4.2 Data of germ-free group

TYPE	n_i	r_i	Data
Thymic lymphoma	29	15	158, 192, 193, 194, 195, 202, 212, 215, 229, 230, 237, 240, 244, 247, 259
Reticulum cell sarcoma	15	8	430, 590, 606, 638, 655, 679, 691, 693
Other causes	37	19	136, 246, 255, 376, 421, 565, 616, 617, 652, 655, 658, 660, 662, 675, 681, 734, 736, 737, 757

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