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DEPENDENCE IN MA MODELS WITH STOCHASTIC PROCESSES

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Abstract. In this paper we present of a class infinite MA(moving-average) sequences of multivariate random vectors. We use the theory of positive dependence to show that in a variety of cases the classes of MA sequences are associated. We then apply the association to establish some probability bounds and moment inequalities for multivariate processes.

1.Introduction

In time series analysis a primary stationary model is the $p \times 1$ moving average (MA) model given by,

$$X(n) = \sum_{j=-\infty}^{\infty} A(j)\epsilon(n-j),\ n = 0, \pm 1, \pm 2, \cdots \tag{1.1}$$

where $A(j), j = 0, \pm 1, \pm 2, \cdots$, is a sequence of $p \times p$ parameter matrices such that $\sum_{j=-\infty}^{\infty} ||A(j)|| < \infty$, and $\epsilon(n), n = 0, \pm 1, \pm 2, \cdots$, is a sequence of uncorrelated $p \times 1$ random vectors with mean zero and common covariance matrix. It is well known that this model emerges from many physically realizable systems(see,for eample,Hannan(1970),p.9).

Gaver and Lewis(1980) consider stationary $ARMA$-type where the random variables $X(n)$ have gamma distributions. Jacobs and Lewis(1983) construct $ARMA$-type models where the random variables $X(n)$ are discrete and assume values in a common finite set. The models mentioned above have been used in various fields of applied probability and time series analysis. In this paper we present a class of infinte MA sequences of multivariate random vectors, has geometric marginals. Within each class of models, the sequences are classified accor ding to their order of dependence on the past. We use the theory of positive dependence to show that

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in a variety of cases the class of MA sequences is associated. We then apply the association to establish some moment and probability inequality. In section 2 we define the multivariate geometric distribution which is the underlying distribution of our paper and the concepts of association and present a variety of multivariate geometric distributions that are associated. In section 3 we costruct a class of MA sequences that has geometric marginals and show that if the distribution is associated,so is the related MA sequence. Finally, in section 4 we underlying relate multivariate point processes to the multivarite geometric MA processes discussed in section 3 and utilize positive dependence properties to obtain some probability bounds and moment inequalities for multivariate processes.

2.Preliminaries

DEFINITION 2.1. *Let* $(X_1, \cdots, X_n)$ *be a random vector assuming values of* X_i*/s in* $\{1, 2, \cdots\}$. *Then* $(X_1, \cdots, X_n)$ *is said to a multivariate geometric distribution* (MVG) *if the* X_i*/s are geometrically distributed. Note that the* $(k-1)$ *dimensional marginals (hence* k*-dimensional marginals,* $k = 1, 2, \cdots, n-1$*) are* MVG.
We introduce some examples multivariate geometrically distributed random variables:
(A) Let $(X_1, \cdots, X_n)$ *be independent geometric. Then* $(X_1, \cdots, X_n)$ *has a multivariate geometric distribution.*
(B) Let $(X_1, \cdots, X_{n+1})$ *be independent geometric random variables and put* $N_1 = \min(X_1, X_{n+1}, \cdots, N_n = \min(X_n, X_{n+1})$. *Then* $(N_1, \cdots, N_n)$ *has multivariate geometric distribution.*
(C) $(M_1, \cdots, M_k)$ *be multivariate geometric and let* $(N_1(j), .., N_k(j))$, $j = 1, 2, \cdots$, *be an i.i.d. sequence of random vectors with multivariate geometric distributions which are independent of* $(M_1, \cdots, M_k)$. *Then* $((N_1(j), \cdots, N_k(j))$ *has a multivariate geometric distribution.*

DEFINITION 2.2. *Let* $X = (X_1, \cdots, X_n)$, $n = 1, 2, \cdots$ *be a multivariate random vector. The random variables* $X_1, \cdots, X_n$ *are called associated if for all pairs of measurable bounded functions* $f, g : R^n \longrightarrow R$ *both nondecreasing in each argument* $Cov(f(X), g(X)) \geq 0$.

The following lemma provides sufficient conditions for some of the multivariate distributions presented in the above examples to be associated.

LEMMA 2.3. *Let* $T = (T_1, \cdots, T_n)$ *be a random vector with components assuming values in the set* $\{1, 2, \cdots\}$ *and let*

$$R(j) = (R_1(j), \cdots, R_n(j)), \quad j = 1, 2, \cdots,$$

be an i.i.d. sequence of nonnegative random vectors independent of T. If $(T_1, \cdots, T_n)$ are associated, and $R_1(j), \cdots, R_n(j)$ are associated, then $\sum_{j=1}^{T_1} R_1(j), \cdots, \sum_{j=1}^{T_n} R_n(j)$ are associated.

Proof. Let $f, g : R^n \longrightarrow R$ be a measurable bounded functions nondecreasing in each argument and let

$$X_1 = \sum_{j=1}^{T_1} R_1(j), \cdots, X_n = \sum_{j=1}^{T_n} R_n(j).$$

First note that

$$\begin{aligned}&Cov(f(X_1, \cdots, X_n), g(X_1, \cdots, X_n))\\&= E(Cov(f(X_1, \cdots, X_n), g(X_1, \cdots, X_n))|T)\\&+ Cov(Ef(X_1, \cdots, X_n)|T, Eg(X_1, \cdots, X_n)|T).\end{aligned}$$

Now, $Ef(X_1, \cdots, X_n)|T$, and $Eg(X_1, \cdots, X_n)|T$ are nondecreasing functions of $T_1, \cdots, T_n$. Since $T_1, \cdots, T_n$ are associated

$$Cov(E[f(X_1, \cdots, X_n)|T], E[g(X_1, \cdots, X_n)|T]) \geq 0.$$

Since $f[(X_1, \cdots, X_n)|T]$ and $g[(X_1, \cdots, X_n)|T]$ are nondecreasing functions of

$R_1(1), \cdots, R_1(T), \cdots, R_n(1), \cdots, R_n(T)$, these random variables are associated (cf. Barlow and Proschan(1975)). Thus

$$Cov(f(X_1, \cdots, X_n)|T, g(X_1, \cdots, X_n)|T) \geq 0.$$

Consequently, $Cov(f(X_1, \cdots, X_n), g(X_1, \cdots, X_n)) \geq 0$ and $X_1, \cdots, X_n$ are associated.

3. Model constructions and Notations

We denote a class of infinite MA sequences by

$$\{G(n,m) = (G_1(n,m), \cdots, G_k(n,m)), n = 0, \pm 1, \pm 2, \cdots\},\ m = 1, 2, \cdots.$$

We show that each $G(n,m)$ has a multivariate geometric distribution with a vector mean independent of n or m. Within each class of sequences the order of dependence on the past is indicated by the parameter m. For each positive integer m, $G(n,m)$ depend only on the previous m variates $\{G(n-1,m),\cdots,G(n-m,m)\}$ where $G(n,\infty)$ depend on all the preceding random vectors $\{G(n-1,\infty),G(n-2,\infty),\cdots\}$. After constructing the various models we presented sufficient conditions for the random variables $\{G_l(n_j,m)\}$, $l=1,2,\cdots,k$; $j=1,2,\cdots,k$ to be associated, where $k=1,2,\cdots$ and $n_1<n_2<\cdots<n_k\in\{0,\pm1,\pm2,\cdots\}$. In this section, we construct the geometric class of sequences:

NOTATION. *Let $p_1,\cdots,p_k$ be real numbers in (0,1] and let $\alpha_1(n),\cdots,\alpha_k(n)$ be sequence of parameters such that $p_j\le\alpha_j(n)\le1$, $j=1,2,\cdots,k$. Further, let $N(n)=(N_1(n),\cdots,N_k(n))$ are independent multivariate geometric vectors with mean vector and let $(p_1^{-1}\alpha_1(n),\cdots,p_k^{-1}\alpha_k(n))$ and let $M_n=(M_1(n),\cdots,M_k(n))$ are i.i.d. multivariate geometrics, independent of all $N(n)$, with mean vector $(p_1^{-1},\cdots,p_k^{-1})$. Finally, let $(J_1(n,j),\cdots,J_k(n,j))$ are independent random vectors, independent of all $M(n)$ and $N(n)$, such that $J_i(n,j)$ is Bernoulli with parameter $(1-\alpha_i(n))$, $i=1,2,\cdots,n$ and let $U_q(n,j)$ be a $n\times n$ random diagonal matrix*

$$U_q(n,j)=diag\{\Pi_{s=q}^{j}J_1(n,k),\cdots,\Pi_{s=q}^{j}J_n(n,s)\},q\in\{1,2,\cdots,j\}.$$

To ease the notation we put $U_1(n,j)=U(n,j)$. We now present the class of geometric sequences. For $m=1,2,\cdots$ and $n=0,\pm1,\pm2,\cdots$ let

$$G(n,m)=\sum_{r=0}^{m}U(n,r)N(n-r)+U(n,m+1)M(n-m) \qquad (*)$$

and

$$G(n,\infty)=\sum_{r=0}^{\infty}U(n,r)N(n-r) \qquad (**)$$

Next, we show that $G(n,m)$ has multivariate geometric distributions.

LEMMA 3.1. *For $n=0,\pm1,\pm2,\cdots$ and $m,q=1,2,\cdots$ let $H_q(n,m)=\sum_{r=0}^{m}U_q(n,r+q-1)N(n-r-q+1)+U_q(n,m+q)M(n-m-q+1)$.*

Then for all n,m, and q, $H_q(n,m)$ has a k-variate geometric distribution with mean vector $(p_1^{-1},\cdots,p_k^{-1})$.

Proof. By an induction argument on m we prove the lemma. For $m=0$,

$$H_q(n,0)=N(n-q+1)+U_q(n,q)M(n-q+1).$$

By computing the characteristic function of the components of $H_q(n,0)$ we can verify that the lemma holds for all n, q. Assume now that the lemma holds for m, and all n, q.

Noting that $H_q(n,m+1) = N(n-q+1) + U_q(n,q)$ $\times[\sum_{r=0}^{m} U_{q+1}(n,r+q)N(n-q-r) + U_{q+1}(n,m+q+1)M(n-m-q)]$, we see that, by induction, the terms in the brackets are k-variate geometric with mean $(p_1^{-1},\cdots,p_k^{-1})$. Since this term is independent of $N(n-q+1)$, it follows as in the case $m=0$ that $H_q(n,m+1)$ has the appropriate distribution for all n and q. Note that $G(n,m)$ given by (**) is equal to $H_1(n,m)$. Thus, we conclude from lemma 3.1 that the following holds.

COROLLARY 3.2. *For all n and m, $G(n,m)$ has a k-variate geometric with mean vector $(p_1^{-1},\cdots,p_k^{-1})$.*

LEMMA 3.3. *For all n, $G(n,\infty)$ has a k-variate geometric distribution with mean vector $(p_1^{-1},\cdots,p_k^{-1})$.*

Proof. Let m be a positive integer. since

$$\lim_{m\to\infty}(1-\alpha_j(n))^m \le \lim_{m\to\infty}(1-p_j)^m = 0, j=1,2,\cdots,k,$$

$$\lim_{m\to\infty} G(n,m) = G(n,\infty)$$

and the results of the lemma follow from corollary 3.2.

LEMMA 3.4. *Suppose that $M_1(1),\cdots,M_k(1)$ are associated and that for all n, $N_1(n),\cdots,N_k(n)$ are associated. Then for all positive integers m, r and all integers $n_1 < n_2 < \cdots < n_r$, the random variables $\{G_i(n_j),m)$, $i=1,\cdots,k\ ;\ j=1,\cdots r\}$ are associated.*

LEMMA 3.5. *Suppose that $M_1(1),\cdots,M_k(1)$ are associated and that for all n, $N_1(n),\cdots,N_r(n)$ are associated. Then for all positive integers k and all integers $n_1 < n_2 < \cdots < n_r$, the random variables $\{G_i(n_j,\infty)$, $i=1,\cdots,k\ ;\ j=1,\cdots,r\}$ are associated.*

Proof. By similar arguments to the ones given in the proof of Lemma 3.3 we conclude that the sequence

$$\{G_1(n_1,m),\cdots,G_k(n_1,m),\cdots,G_1(n_r,m),\cdots,G_k(n_r,m)\}$$

converges in distribution as $m\to\infty$ to

$$\{G_1(n_1,\infty),\cdots,G_k(n_1,\infty),\cdots,G_1(n_r,\infty),\cdots,G_k(n_r,\infty)\}.$$

By Lemma 3.4, the $G_i(n_j,m)$, $i=1,2,\cdots,k$, $j=1,\cdots,r$ are associated for all m. Consequently, the results of the lemma follow by P_4 Esary et al(1967).

4.Probability Inequalities

Throughout this section we fixed $m = 1, 2, \cdots, \infty$ and hence suppress it from our notation, that is, $G(n, m)$ is denoted by $G(n)$.

In the point process theory of the models, the behavior of the vector of sums

$$T_G(r) = (T_{G_1}(r_1), \cdots, T_{G_k}(r_k))$$

where

$$T_{G_i}(r_i) = \sum_{n=1}^{r_i} G_i(n), i = 1, 2, \cdots, k$$

is of interest, $r_1, \cdots, r_k \in \{1, 2, \cdots\}$. For example, if $G(n)$ is a vector of k-variate geom etric wating times of a count process

$$N_G(r) = (N_{G1}(r_1), \cdots, N_{Gk}(r_k))$$

which are the number of occurences by trials $r_1, \cdots, r_k \in \{1, 2, \cdots\}$, then

$$N_{G_i}(r_i) = T_{G_i}(r_i),\ i = 1, 2, \cdots, k.$$

We now utilize positive dependence properties to obtain probability bounds for sums $T_G(r)$ and moment inequalities for the process $G(n)$. First, we define concepts of positive dependence.

DEFINITION 4.1. *Let $k = 2, 3, \cdots$ and let $X = (X_1, \cdots, X_k)$ be a random vector. X is said to be positively upper orthant dependent(PUOD) (positively lower orthant dependent(PLOD)) if for all real numbers $t_1, \cdots,$ t_k,*

$$P(X_i > t_i : i = 1, \cdots, k) \geq \Pi_{i=1}^k P(X_i > t_i) \tag{4.1}$$

$$P(X_i \leq t_i : i = 1, \cdots, k) \geq \Pi_{i=1}^k P(X_i \leq t_i) \tag{4.2}$$

Moreover, the random vector $X = (X_1, \cdots, X_k)$ be positively orthant depent(POD) if they satisfy both PUOD and PLOD

REMARK. *(A) If $X = (X_1, \cdots, X_k)$ be associated then X is PUOD and PLOD. (B) Let $f_1, \cdots, f_k : (-\infty, \infty) \longrightarrow [0, \infty)$ be measurable nondecreasing (nonincreasing) functions and let X be PUOD(PLOD). Then*

$$E\Pi_{i=1}^k f_i(X_i) \geq \Pi_{i=1}^k Ef_i(X_i)$$

(see Lehmann (1966)).

LEMMA 4.2. *Suppose that for* $q = 1, 2, \cdots$ *the random variables* $\{G_i(n), i = 1, 2, \cdots, k; n = 1, 2, \cdots, q\}$ *are associated.*
Then for $r_1, \cdots, r_k \in \{1, 2, \cdots, k\}$ $\{T_{G_i}(r_i), i = 1, 2, \cdots$ *is associated.*

Proof. It follows from the fact that $\{T_{G_i}(r_i), i = 1, 2, \cdots, k\}$ are non-decreasing functions of associated random variables.

Next, we obtain the following lemma for the sum $T_G(r)$.

LEMMA 4.3. *Assume that* $\alpha_1(n), \cdots, \alpha_k(n)$ *are equal to* $\alpha_1, \cdots, \alpha_k$, *respectively, for all* n. *Let* $NB_i(r_i, \theta_i), i = 1, 2, \cdots, k$ *be negative multinomial random variables with parameters* (r_i, θ_i). *Then*

$$T_{G_i}(r_i) \geq NB_i(r_i, p_i\alpha_i^{-1}), i = 1, 2, \cdots, k.$$

If in addition, the random variables $\{G_i(n), i = 1, 2, \cdots, k, n = 1, 2, \cdots, q\}$, $q = 1, 2, \cdots$, *are associated, then for* $a_1 \geq r_1, \cdots, a_k \geq r_k$, $P(T_{G_1}(r_1) \geq a_1, \cdots, T_{G_k}(r_k) \geq a_k)$

$$\geq P(NB_1(r_1, p_1\alpha_1^{-1}) \geq a_1) \cdots P(NB_k(r_k, p_k\alpha_k^{-1} \geq a_k) \tag{4.3}$$

Proof. From the equations $(*)$ or $(**)$ we see that $G_i(n) \geq N_i(n)$, $i = 1, \cdots, k; n = 1, 2, \cdots$. Hence

$$P(T_{G_i}(r_i) \geq a_i) = P(\sum_{n=1}^{r_i}(G_i(n) \geq a_i))$$

$\geq P(\sum_{n=1}^{r_i}(N_i(n) \geq a_i)) = P(NB_i(r_i, p_i\alpha_i^{-1}) \geq a_i), i = 1, 2, \cdots, k$
and the first assertion is proved. Since $T_{G_1}(r_1), \cdots, T_{G_k}(r_k)$ are associated they are PUOD. (4.3) is obtained.

LEMMA 4.4. *Assume that the random variables* $\{G_i(n), i = 1, 2, \cdots, k; n = 1, 2, \cdots, h\}$, $h = 1, 2, \cdots$ *are associated. Let* $k_1, \cdots, k_q$ *are positive integers and let* $l_1, \cdots, l_q \in \{1, 2, \cdots k\}; q = 1, 2, \cdots$ *Then*

$$E\Pi_{j=1}^{q}(G_{l_j}(j))^{k_j} \geq \Pi_{j=1}^{q} E(G_{l_j}(1)^{k_j}).$$

Proof. The result follows from random variables $G_{l_j}^{-1}(n)$ and corollary 3.2

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