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ON A CLASS OF UNIVALENT FUNCTIONS

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Abstract

For A and B, $-1 \leq B < A \leq 1$, let $P[A,B]$ be the class of functions p analytic in the unit disk E with $P(0) = 1$ and subordinate to $\frac{1+Az}{1+Bz}$. We introduce the class $T_\alpha[A,B]$ of functions $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in E and for $z \in E$, $\alpha \geq 0$, $[(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)] \in P[A,B]$. It is shown that, for $\alpha \geq 1$, $T_\alpha[A,B]$ consists entirely of univalent functions and the radius of univalence for $f \in T_\alpha[A,B]$, $0 < \alpha < 1$ is obtained. Coefficient bounds and some other properties of this class are studied. Some radii problems are also solved

1. Introduction

Let $\mathcal{A}$ denote the class of the function f:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. Also let S, K, S^* and C denote the classes of all functions in $\mathcal{A}$ which are, respectively, univalent, close-to-convex, starlike and convex in E.

The class $P[A,B]$ is defined by Janowski [4] as follows:

Let p be a function of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n, \tag{1.2}$$

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which is analytic in E. Then $p \in P[A,B]$ if and only if, for $z \in E$, $-1 \le B < A \le 1$,

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)},$$

where w is a Schwarz function which is analytic in E, $w(0) = 0$, $|w(z)| < 1$ for all $z \in E$.

Special selections of A and B lead to familiar sets defined previously [4]. In particularly $P[A,B] \subset P[1,-1] = P$, the class of functions with positive real part.

To avoid repetition we lay down, once for all, that $-1 \le B < A \le 1$ and $z \in E$, unless mentioned otherwise.

A univalent function $f \in S^*[A,B]$ if and only if $\frac{zf'(z)}{f(z)} \in P[A,B]$. Clearly $S^*[A,B] \subset S^*[1,-1] = S^*$, the class fo starlike functions of order σ defined by Robertson in [9]. We define the following.

DEFINITION 1.1. Let $f \in \mathcal{A}$. Then f is said to belong to the class $T_\alpha[A,B]$ if and only if, for $z \in E$, $\alpha \ge 0$,

$$[(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)] \in P[A,B].$$

For special values of α, A and B, we obtain various previously known classes of analytic functions, e.g. see [3].

Let f and g be in $\mathcal{A}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. We define the convolution operator $\Gamma : \mathcal{A} \to \mathcal{A}$ by $\Gamma(g) = f * g$ for given $f \in \mathcal{A}$, where

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Ruscheweyh [11] has defined the class B_σ of prestarlike functions of order σ as follows.

DEFINITION 1.2. A function $f \in \mathcal{A}$ belong to the class B_σ of prestarlike functions of order σ if and only if, for $z \in E$,

$$Re\frac{f(z)}{zf'(0)} > \frac{1}{2}, \quad \text{for } \sigma = 1$$

and

$$\frac{z}{(1+z)2(1-\sigma)} * f(z) \in S^*(\sigma), \quad \text{for } 0 \le \sigma < 1.$$

2. Preliminary Lemmas

We give here some basic results which we shall need later on. For proof of both, we refer to [13].

LEMMA 2.1. *For $\sigma \leq 1$, let $f \in B_\sigma$, $g \in S^*(\sigma)$ and F analytic in E. Then the generalized convolution operator*

$$\Lambda F = \frac{f * gF}{f * g}$$

is a convexity preserving operator.

LEMMA 2.2. *If p is analytic in E, $p(0) = 1$ and $Rep(z) > \frac{1}{2}$, $z \in E$, then for any function F, analytic in E, the function $p * F$ takes values in the convex hull of the image of E under F.*

LEMMA 2.3. *[4] If $p \in P[A, B]$. Then for $z \in E$, $|z| = r$, we have*

$$Re\frac{zp'(z)}{p(z)} \geq -\frac{(A-B)r}{(1-Ar)(1-Br)}, \quad \text{for } r \leq r_1,$$

where r_1 is the unique root of the equation

$$ABr^4 - 2ABr^3 + (2A + 2B - 1)r^2 - 2r + 1 = 0.$$

in the interval (0, 1].

3. Main Results

We now proceed to investigate the class $T_\alpha[A, B]$.
Let, for $\alpha > 0$

$$k(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z \frac{t^{\frac{1}{\alpha}-1}}{1-t} dt \tag{3.1}$$

The function k is convex, see [10].
We note here that the function $F \in T_\alpha[A, B]$, $\alpha > 0$ can be obtained by taking the Hadamard product (convolution), of the convex function k with the function s, where $s(z) = zp(z)$, $p \in P[A, B]$.

THEOREM 3.1. *Let $f \in T_\alpha[A,B]$, $\alpha > 0$ and be given by (1.1). Then for $n \geq 2$,*

$$|a_n| \leq \frac{A-B}{1-\alpha+\alpha n}.$$

This result is sharp.

Proof. Since $f \in T_\alpha[A,B]$, we have

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = p(z), \quad p \in P[A,B], \tag{3.2}$$

Let p be given by $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. It is known [1] that

$$|c_n| \leq (A-B), \quad n \geq 1, \tag{3.3}$$

Equating the coefficients of z^{n-1} in (3.2) and using (3.3), we obtain the required result.

Sharpness, for each n, follows from the functions f_n:

$$f_n(z) = k(z) * zp_n(z),$$

where $k(z)$ is given by (3.1) and

$$P_n(z) = \frac{1+A\delta z^n}{1+B\delta z^n}, \quad |\delta| = 1,$$

THEOREM 3.2. *The class $T_\alpha[A,B]$ is a convex set.*

Proof. Let $f, g \in T_\alpha[A,B]$ and let, for $0 \leq \lambda < 1$,

$$F(z) = \lambda f(z) + (1-\lambda)g(z).$$

Then

$$\begin{aligned}(1-\alpha)\frac{F(z)}{z} + \alpha F'(z) &= (1-\alpha)\left[\lambda\frac{f(z)}{z} + (1-\lambda)\frac{g(z)}{z}\right] \\ &\quad + \alpha[\lambda f'(z) + (1-\lambda)g'(z)] \\ &= \lambda p_1(z) + (1-\lambda)p_2(z) = p(z),\end{aligned}$$

where $p_1, p_2 \in P[A,B]$ and since $P[A,B]$ is a convex set, see [8], so $p \in P[A,B]$ and hence $F \in T_\alpha[A,B]$.

THEOREM 3.3. *Let $\phi \in C$ and $f \in T_\alpha[A,B]$, $\alpha \geq 0$. Then $\phi * f \in T_\alpha[A,B]$.*

Proof. Let $G = \phi * f$. Then, for $z \in E$,

$$(1-\alpha)\frac{G(z)}{z} + \alpha G'(z) = (1-\alpha)\frac{(\phi * f)(z)}{z} + \alpha(\phi * f)'(z)$$
$$= \frac{\phi(z)}{z} * \left[(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right]$$

Since ϕ is convex, we have $Re\frac{\phi(z)}{z} > \frac{1}{2}$ for $z \in E$. Thus, using Lemma 2.2 and the fact that $f \in T_\alpha[A,B]$, we obtain the required result that

$$G = \phi * f \in T_\alpha[A,B].$$

From the proof of theorem 3.3, it is clear that, in fact, the following more general result holds.

THEOREM 3.4. *If $g \in \mathcal{A}$ with $Re(\frac{g(z)}{z}) > \frac{1}{2}$, $z \in E$ and $f \in T_\alpha[A,B]$, then $(f * g) \in T_\alpha[A,B]$.*

COROLLARY 3.1. *Let $f \in T_\alpha[A,B]$, $\alpha \geq 0$. Then $T_\alpha[A,B]$ is invariant under the following integral operators.*

(i) $$f_1(z) = \int_0^z \frac{f(t)}{t}dt.$$

(ii) $$f_2(z) = \frac{2}{z}\int_0^z f(t)dt. \quad \textit{(Libera's operator [5])}$$

(iii) $$f_3(z) = \int_0^z \frac{f(t) - f(xt)}{t - xt}dt, \quad |x| \leq 1,\ x \neq 1$$

(iv) $$f_4(z) = \frac{1+c}{z^c}\int_0^z t^{c-1}f(t)dt,\ Re c > 0$$

Proof. We may write, see [2],

$$f_1(z) = f(z) * \phi_1(z),$$
$$f_2(z) = f(z) * \phi_2(z),$$
$$f_3(z) = f(z) * \phi_3(z),$$
$$f_4(z) = f(z) * \phi_4(z),$$

where ϕ_i, $i = 1, 2, 3, 4$ are convex and

$$\phi_1(z) = -\log(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n,$$

$$\phi_2(z) = \frac{-2[z+\log(1-z)]}{z} = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n,$$

$$\phi_3(z) = \frac{1}{1-x} \log\left[\frac{1-xz}{1-z}\right] = \sum \frac{1-x^n}{n(1-x)} z^n, \ |x| \leq 1, \ x \neq 1,$$

$$\phi_4(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \ Re c > 0$$

The result follows by applying theorem 3.3.

REMARK 3.1. Since $T_\alpha[A, B] \subset T_\alpha[1, -1] = T_\alpha$ and T_α is known [3] to consist of univalent functions for $\alpha \geq 1$, it follows that $f \in T_\alpha[A, B]$, $\alpha \geq 1$ is univalent.

We now prove a covering theorem.

THEOREM 3.5. *Let $f \in T_\alpha[A, B]$, $\alpha \geq 1$. If D is the boundary of the image of E under f, then every point of D has a distance of at least $\frac{(1+\alpha)}{[2(1+\alpha)+(A-B)]}$ from the origin.*

Proof. Let $f(z) \neq c$, $c \neq 0$. Hence $f_1(z) = cf(z)/(c - f(z))$ is univalent in E. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then

$$\frac{cf(z)}{c - f(z)} = \frac{cz + ca_2 z^2 + \cdots}{c - z - a_2 z^2 - \cdots}$$
$$= z + (a_2 + \frac{1}{c})z^2 + \cdots,$$

and since f_1 is univalent, it follows that

$$|a_2 + \frac{1}{c}| \leq 2.$$

Hence

$$|1/c| \leq 2 + |a_2|,$$

and using theorem 3.1 for $n = 2$, we have

$$|\frac{1}{c}| \leq 2 + \frac{A-B}{1+\alpha} = \frac{2(1+\alpha)+(A-B)}{1+\alpha}.$$

Thus

$$|c| \geq \frac{(1+\alpha)}{2(1+\alpha)+(A-B)}$$

We shall now prove some radii problems for the class $T_\alpha[A,B]$.

THEOREM 3.6. *Let $f \in T_\alpha[A,B]$, $0 < \alpha < 1$. Then $f \in T_1[A,B]$ and hence univalent for $|z| < r_0$, where r_0 is the radius of the largest disk centered at the origin for which $Rek'(z) > \frac{1}{2}$, $k(z)$ is defined by (3.1) and r_0 is given by the smallest positive root of the equation*

$$\text{(3.4)} \qquad \frac{(2/\alpha)-1-r}{1+r} - \frac{2}{\alpha}(\frac{1}{\alpha}-1)\int_0^1 \frac{\xi^{\frac{1}{\alpha}-1}}{1+\xi r}d\xi = 0.$$

This result is sharp.

Proof. Since $f \in T_\alpha[A,B]$, we may write

$$f(z) = z(k(z) * \int_0^z p(t)dt)', \quad p \in P[A,B]$$

and k is defined by (3.1).

Thus

$$f'(z) = \frac{zp(z) * zk'(z)}{z} = \frac{zp(z) * zk'(z)}{z * zk'(z)}$$

Let

$$zk'(z) = h(z).$$

Then

$$h'(z) = k'(z) + zk''(z).$$

It is easy to see that $k'(0) = 1$. Therefore, for $Rek'(z) > \frac{1}{2}$, we see that

$$Re\frac{h(z)}{zh'(0)} > \frac{1}{2}$$

in $|z| < r_0$. Hence h is a prestarlike function of order $\sigma = 1$. Also, since $g(z) \equiv z \in S^*$, we can apply Lemma 2.1 and hence $f \in T_1[A,B]$ for $|z| < r_0$.

To find radius r_0, we proceed as follows.

From (3.1), we have

$$k'(z) = \frac{1}{\alpha(1-z)} - \frac{1}{\alpha}(\frac{1}{\alpha} - 1)z^{\frac{-1}{\alpha}} \int_0^z \frac{t^{\frac{1}{\alpha}-1}}{1-t}dt. \tag{3.5}$$

Powers in (3.5) are meant as principal values. The function k' is analytic in E, $k'(0) = 1$ and

$$2k'(z) - 1 = \frac{2-\alpha+\alpha z}{\alpha(1-z)} - \frac{2}{\alpha}(\frac{1}{\alpha} - 1)\int_0^1 \frac{\xi^{\frac{1}{\alpha}-1}}{1+\xi r}d\xi.$$

Therefore $Re k'(z) > \frac{1}{2}$ for $|z| < r_0$, where r_0 is the smallest positive root of (3.5).

The function $f_0 \in T_\alpha[A,B]$, defined as

$$f_0(z) = \frac{z(1+Az)}{1+Bz} * k(z)$$

shows that the above result is sharp, since $f_0'(r_0) = 0$.

REMARK 3.2.

$f \in T_0[A,B] \subset T_0[1,-1]$ is univalent in $|z| < \sqrt{2} - 1$, see [7].

THEOREM 3.7. *Let $f \in T_0[A,B]$. Then $f \in T_\alpha[A,B]$ for $|z| < r_\alpha$, $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$, where*

$$r_\alpha = \frac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}}, \tag{3.6}$$

This radius is best possible,

Proof. Let

$$\phi_\alpha(z) = (1-\alpha)\frac{f(z)}{z} + \alpha f'(z)$$

Then

$$\phi_\alpha(z) = \left(\frac{k_\alpha(z)}{z}\right) * \left(\frac{f(z)}{z}\right),$$

where

$$k_\alpha(z) = (1-\alpha)\frac{z}{1-z} + \alpha\frac{z}{(1-z)^2}$$
$$= z + \sum_{n=1}^{\infty}(1+(n-1)\alpha)z^n.$$

It is clear that k_α is convex for $|z| < r_\alpha$, where r_α is given by (3.6) and this radius is sharp. Consequently, for $|z| < r_\alpha$, $Re\frac{k_\alpha(z)}{z} > \frac{1}{2}$, see [14]. Hence, since $\frac{f(z)}{z} \in P[A,B]$, we apply Lemma 2.2 to obtain the required result.

Next we find the radius of starlikeness for $f \in T_\alpha[A,B]$.

THEOREM 3.8. *Let $f \in T_\alpha[A,B]$, $A > 1/(2-B)$. Then $f \in S^*$ for $|z| < R_0$ where $R_0 = \min(r_1, r_2), r_1$ is the unique root of the equation (2.1) in the interval (0, 1] and r_2 is given by*

$$r_2 = \frac{1}{A+\sqrt{A^2-AB}}$$

This result is sharp.

Proof.

$$f \in T_\alpha[A,B] \text{ implies that}$$

$$f(z) = k(z) * zp(z), \quad p \in P[A,B]$$

and k is given by (3.1).

Let $s(z) = zp(z)$.

Then

$$Re\frac{zs'(z)}{s(z)} = 1 + Re\frac{zp'(z)}{p(z)},$$

and on using Lemma 2.3, we have

$$Re\frac{zs'(z)}{s(z)} \geq \frac{1-2Ar+ABr^2}{(1-Ar)(1-Br)}, \quad \text{for } r \leq r_1.$$

Let $T(r) = 1-2Ar+ABr^2$. Then $T(0) = 1 > 0$ and $T(1) = 1-2A+AB < 0$ if $A > \frac{1}{2-B}$. Therefore $T(r)$ has at least one root in (0,1). Let r_2 be the smallest positive root of $T(r) = 0$ less than one. Hence $zp \in S^*$ for $|z| < R_0 = \min(r_1, r_2)$.

Since $k \in c$ for $z \in E$, we reduce that $f \in S^*$ for $|z| < R_0$ where we have used a result due to Ruscheweyh and Sheil-Small [12].

Sharpness follows from the function

$$f(z) = k(z) * zp_e(z),$$

$$p_e(z) = \frac{1+Az}{1+Bz}.$$

We note that, for $A = 1, B = -1, r_0 = \sqrt{2} - 1$. Let μ_1 and μ_2 be linear operators defined on the class A as follows

$\mu_1(f(z)) = zf'(z)$,

$\mu_2(f(z)) = [f(z) + zf'(z)]/2$. (Livingston's operaotr [6]).

Both of these operators can ve written as a convolution operator [2] given by

$$\mu_i(f) = h_i * f, \quad i = 1, 2,$$

where

$$h_1(z) = \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2},$$

$$h_2(z) = \sum_{n=1}^{\infty} \frac{n+1}{2} z^n = \frac{z - \frac{z^2}{2}}{(1-z)^2}.$$

It can easily be verified that the radius of convexity $r_c(h_1) = 2 - \sqrt{3}$ and $r_c(h_2) = \frac{1}{2}$. These facts together with theorem 3.3 yield the following.

THEOREM 3.9. *Let $f \in T_\alpha[A, B]$, $\alpha \geq 0$. Then $\mu_1(f) = f * h_1 \in T_\alpha[A, B]$ for $|z| < 2 - \sqrt{3}$, and $\mu_2(f) = f * h_2 \in T_\alpha[A, B]$ for $|z| < \frac{1}{2}$.*

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