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CHARACTERIZATIONS OF COMMUTATIVE BCI–ALGEBRAS

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In [4], J. Meng and X. L. Xin introduced the concept of commutative BCI-algebras and discussed the structure of such algebras. The aim of this paper is to obtain a characterization of commutative BCI-algebras.

Recall that a BCI-algebra is a non-empty set X with a binary operation $*$ and a constant 0 satisfying the axioms

(1) $(x*y)*(x*z) \leq z*y$,
(2) $x*(x*y) \leq y$,
(3) $x \leq x$,
(4) $x \leq y$ and $y \leq x$ imply $x = y$,
(5) $x \leq 0$ implies $x = 0$,

where $x \leq y$ is defined by $x*y = 0$. Further if $x \geq 0$ for all x, then X is a BCK-algebra. Any BCK-algebra is a BCI-algebra [2]. A BCK-algebra is commutative if it satisfies the identity $x*(x*y) = y*(y*x)$. In this case $y*(y*x) = x \wedge y$, the greatest lower bound of x and y.

In a BCI-algebra the following properties hold:

(6) $(x*y)*z = (x*z)*y$.
(7) $x*(x*(x*y)) = x*y$.

DEFINITION 1 ([4]). A BCI-algebra X is said to be commutative if whenever $x \leq y$ then $x = y*(y*x)$ for all $x, y \in X$.

DEFINITION 2 ([1]). Let X be a BCI-algebra and let $x \in X$. Then the set

$$A(x) = \{y \in X | y \leq x\}$$

is called the initial section of x.

DEFINITION 3 ([3]). An element a of a BCI-algebra X is called an atom if $z*a = 0$ implies $z = a$ for all $z \in X$. The set of all atoms of X is denoted by $L(X)$. For any atom a of X, the set $V(a) = \{x \in X | a \leq x\}$ is called a branch of X.

Obviously $0 \in L(X)$ and $V(0) = X_+$, the p-radical of X.

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LEMMA 1 ([2]). *A BCI-algebra in which $x*(x*y) = y*(y*x)$ holds for any x, y is a BCK-algebra.*

LEMMA 2 ([4]). *Let X be a commutative BCI-algebra. Then for any atom a and all x and y of $V(a)$, we have $x*(x*y) = y*(y*x)$.*

THEOREM 1. *Let X be a BCI-algebra satisfying $A(x) \cap A(y) = A(x \wedge y)$ for all $x, y \in V(a), a \in L(X)$. Then X is a commutative BCK-algebra.*

Proof. Note that $A(x \wedge y) = A(y \wedge x)$ for every $x, y \in V(a), a \in L(X)$. Since $A(x) = A(y)$ if and only if $x = y$, it follows that $x \wedge y = y \wedge x$, i.e., $x*(x*y) = y*(y*x)$. Hence by Lemma 1, X is a commutative BCK-algebra.

THEOREM 2. *If a BCI-algebra X is commutative then for all $x, y \in V(a), a \in L(X)$, we have $A(x) \cap A(y) = A(x \wedge y)$.*

Proof. Let $r \in A(x) \cap A(y)$; then $r \leq x$ and $r \leq y$. Since X is commutative, $r \leq y$ implies $r = y*(y*r)$. Hence

$$\begin{aligned} r*(x \wedge y) &= r*(y*(y*x)) \\ &= (y*(y*r))*(y*(y*x)) \\ &\leq (y*x)*(y*r) \\ &\leq r*x \\ &= 0. \end{aligned}$$

It follows that $r*(x \wedge y) = 0$, i.e., $r \leq x \wedge y$. Thus $r \in A(x \wedge y)$, showing that $A(x) \cap A(y) \subseteq A(x \wedge y)$. To prove the reverse inclusion, let $r \in A(x \wedge y)$. Then $r \leq x \wedge y$. Since $x \wedge y = y*(y*x) = x*(x*y)$ by Lemma 2, it follows from (2) that $x \wedge y \leq x$ and $x \wedge y \leq y$. As the relation $\leq$ is transitive, we have $r \leq x$ and $r \leq y$, and so $r \in A(x)$ and $r \in A(y)$. This means that $r \in A(x) \cap A(y)$, so that $A(x \wedge y) \subseteq A(x) \cap A(y)$.

Since any commutative BCK-algebra is a commutative BCI-algebra [4], we have a characterization of commutative BCI-algebras.

COROLLARY 1. *A BCI-algebra X is commutative if and only if*

$$A(x) \cap A(y) = A(x \wedge y)$$

for all $x, y \in V(a), a \in L(X)$.

REMARK. In a non-commutative BCI-algebra X, the following result is not, in general, true:

(8) $x \leq z$ and $z * y \leq z * x$ imply $x \leq y$

for any $x, y, z \in X$, as shown in the following example.

EXAMPLE. Let $X = \{0, a, b, c, d, e, f, g\}$ and $*$ is defined by

*	0	a	b	c	d	e	f	g
0	0	0	0	0	d	d	d	d
a	a	0	0	0	e	d	d	d
b	b	b	0	0	f	f	d	d
c	c	b	a	0	g	f	e	d
d	d	d	d	d	0	0	0	0
e	e	d	d	d	a	0	0	0
f	f	f	d	d	b	b	0	0
g	g	f	e	d	c	b	a	0

Then X ia a non-commutative BCI-algebra which does not satisfy (8), because $e \leq f$ but $e \neq f * (f * e)$, and $e \leq f$ and $f * d \leq f * e$ but $e * d \neq 0$.

We now give a characterization of commutative BCI-algebras.

THEOREM 3. *A BCI-algebra X is commutative if and only if it satisfies*

(8) $x \leq z$ *and* $z * y \leq z * x$ *imply* $x \leq y$

for all $x, y, z \in X$.

Proof. Let X be a commutative BCI-algebra such that $x \leq z$ and $z * y \leq z * x$ for all $x, y, z \in X$. Then we have $x = z * (z * x)$. Hence $x * y = (z * (z * x)) * y = (z * y) * (z * x) = 0$, which implies that $x \leq y$.

Conversely assume that $x \leq z$ and $z * y \leq z * x$ imply $x \leq y$ for all $x, y, z \in X$. Let $u \leq v$ for all $u, v \in X$. It is sufficient to show that $v*(v*(v*u)) \leq v*u$. But this is obvious by (7). Hence X is a commutative BCI-algebra.

L. H. Shi [5] characterized BCI-algebras as follows.

LEMMA 3. *Let X be an abstract algebra of type (2, 0) with a binary operation $*$ and a constant 0. X is a BCI-algebra if and only if it satisfies the following conditions:*

(1) $(x * y) * (x * z) \leq z * y$,

(4) $x \leq y$ *and* $y \leq x$ *imply* $x = y$,

(9) $x * 0 = x$.

By Lemma 3 and Theorem 3, a commutative BCI-algebra is characterized as follows:

THEOREM 4. *An algebra X of type (2, 0) is a commutative BCI-algebra if and only if it satisfies:*

(1) $(x * y) * (x * z) \leq z * y$,
(4) $x \leq y$ and $y \leq x$ *imply* $x = y$,
(9) $x * 0 = x$,
(8) $x \leq z$ and $z * y \leq z * x$ *imply* $x \leq y$.

References

1. M. A. Chaudhry, *Branchwise commutative BCI-algebras*, Math. Japon. **37** (1992), 163–170.
2. K. Iséki, *On BCI-algebras*, Math. Seminar Notes(presently Kobe J. Math.) **8** (1980), 125–130.
3. J. Meng and X. L. Xin, *Characterization of atoms in BCI-algebras*, Math. Japon. **37** (1992), 359–362.
4. J. Meng and X. L. Xin, *Commutative BCI-algebras*, Math. Japon. **37** (1992), 569–572.
5. L. H. Shi, *An axiom system of BCI-algebras*, Math. Japon. **30** (1985), 351–352.