Honam Mathematical J. 15(1993), No. 1,pp. 47–53

AN EXAMPLE OF SUBREGULAR GERMS FOR 4×4 SYMPLECTIC GROUPS*

YANG-KOHN KIM

Dept. of Mathematics, Chonbuk National University, Chonbuk 560-765, Korea.

We found Shalika's unipotent regular germs in the case of $G=Sp_4(F)$ for p-adic fields F. Next, subregular germs were also found in part for a particular elliptic torus.

The rest of these have to be found explicitly, although they are not so easily obtainable as before.

0. Introduction

Assume that G is the set of F-points of a connected semi-simple algebraic group defined over a p-adic field F with its ring of integers A whose maximal ideal is P with residual characteristic greater than 2, that T is a Cartan subgroup of G, and that T' denote the set of regular elements in T. Letting $d\dot{g}$ be a G-invariant measure on the quotient space $T\backslash G$ and $C_c^\infty(G)$ be the set of smooth functions, we know that for any $f \in C_c^\infty(G)$ and $t \in T'$ the orbital integral $\int_{T\backslash G} f(g^{-1}tg)d\dot{g}$ is convergent. Now, let Su be the set of unipotent conjugacy classes in G and dx_0 be a G-invariant measure on $0 \in Su$. It is also known that $\Lambda_0(f) = \int_0 f dx_0$ is convergent for any $f \in C_c^\infty(G)$.

Shalika's theorem (see [15], p. 236) says that for any $t \in T'$ sufficiently colse to the identity there exist germs $\Gamma_0(t)$ satisfying

$$\int_{T\backslash G} f(g^{-1}tg)d\dot{g} = \sum_{0\in Su} \Gamma_0(t)\Lambda_0(f).$$

Shalika J. A., Howe R., Harish Chandra, Rogawski J., and others contributed to the establishment of the germ associated to the trivial unipotent class.

Received March 31, 1993.

*This paper was supported (in part) by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1991.

Recently Repka J. has found regular and subregular germs for p-adic $GL_n(F)$ and $SL_n(F)$. [5] and [6] say about the regular germs for p-adic $Sp_4(F)$. [7] says about a part of subregular germs in p-adic $Sp_4(F)$.

Now this paper intends to find the rest of these subregular germs. Most conventions shall be those used in [7].

1. Unipotent Orbits in $Sp_4(F)$.

$G = Sp_4(F)$ acts on itself by conjugation, thus in particular on the set of all unipotent elements in G. Refering to [5] §3, we may obtain the following.

PROPOSITION 1.0. *Any unipotent orbit meet the set of all elements of the form*

$$\begin{pmatrix} 1 & x & \alpha & \beta \\ 0 & 1 & \beta-\gamma x & \gamma \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \qquad \text{with } \alpha, \beta, \gamma \text{ and } x \in F. \tag{1.1}$$

If $x \neq 0$ in (1.1), we see that the associated unipotent orbits meet the set of non-regular unipotent matrices or the set of regular unipotent matrices which as a G-set has representatives of the form

$$\begin{pmatrix} 1 & 1 & 0 & \bar{a} \\ 0 & 1 & 0 & \bar{a} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \qquad \text{with } \bar{a} \in F^\times/(F^\times)^2. \tag{1.2}$$

If $x = 0$, it is not a regular unipotent element. Hence (1.2) represents the orbits of the G-set consisting of all the regular unipotent elements of G. According to [7], the subregular unipotent matrices are represented by the form

$$\bar{u}(\bar{\alpha}, \bar{\gamma}) = \begin{pmatrix} 1 & 0 & \bar{\alpha} & 0 \\ 0 & 1 & 0 & \bar{\gamma} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{with } \bar{\alpha}, \bar{\gamma} \in F^\times/(F^\times)^2 \text{ and } |\bar{\alpha}| \geq |\bar{\gamma}| \geq 1,$$

some of which are conjugate. The number of subregular conjugacy calsses ranges from 6 to 8.

Now let $\bar{S}(\bar{\alpha},\bar{\gamma}) = \{ g \in K : g \equiv \bar{u}(\bar{\alpha},\bar{\gamma}) \mod P \}$. Any element of $\bar{S}(\bar{\alpha},\bar{\gamma})$ is conjugate to the form

$$(1.3) \qquad \begin{pmatrix} 1+z_{11} & z_{12} & \bar{\alpha} & 0 \\ \frac{\bar{\gamma}}{\bar{\alpha}} z_{12} & 1+z_{22} & 0 & \bar{\gamma} \\ \frac{1}{\bar{\alpha}} z_{11} & \frac{1}{\bar{\alpha}} z_{12} & 1 & 0 \\ \frac{1}{\bar{\alpha}} z_{12} & \frac{1}{\bar{\gamma}} z_{22} & 0 & 1 \end{pmatrix} \qquad \text{for some } z_{ij} \in P.$$

For later purpose, we let $\bar{S}_3(\bar{\alpha},\bar{\gamma})$ be the set of all such last matrices of the form (1.3), and $\hat{P}$ be the composite map of conjugations which make any element of $\bar{S}(\bar{\alpha},\bar{\gamma})$ reach to the form (1.3).

2. Determining Integrands for subregular germs.

According to [7] Proposition (3.0), we see that for representative pairs $\bar{x} = (\bar{\alpha},\bar{\gamma})$ with $-\frac{\bar{\alpha}}{\bar{\gamma}} \notin (F^\times)^2$ the unipotent orbit of $\bar{u}(\bar{x})$ intersect $\bar{S}(\bar{y})$ if and only if $\bar{x} = \bar{y}$. Now assume θ to be a nonsquare in $F^\times$ and put $E^\theta = F(\sqrt{\theta})$. Let $E_1^\theta = \{ a + b\sqrt{\theta} : a, b \in F \text{ and } a^2 - \theta b^2 = 1 \}$.

Assuming that T is the set of all matrices of the form

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & \alpha & 0 & \beta \\ b\theta_1 & 0 & a & 0 \\ 0 & \beta\theta_2 & 0 & \alpha \end{pmatrix}$$

with $\theta_j \in F^\times \backslash (F^\times)^2$ and $a^2 - b^2\theta_1 = \alpha^2 - \beta^2\theta_2 = 1$, we see easily that T as an elliptic torus is isomorphic to $E_1^{\theta_1} \times E_1^{\theta_2}$ both algebraically and topologically. According to the Shalika's theorem as we have mentioned before, we have a kind of expansion

$$(2.0) \qquad \int_{T\backslash G} f(t^g)d\dot{g} = \sum_{j=1}^{n} \Gamma_j(t) \int_{Z(u_j)\backslash G} f(u_j^g)d\dot{g},$$

where $\{ u_j \}$ is a finite set of representatives of the unipotent orbits, $f \in C_c^\infty(G)$, and t is any regular element $t \in T'$ sufficiently close to the id. Here the functions Γ_j called Shalika's germs do not depend on f, but depend on a maximal torus T. We want to calculate the functions $\Gamma_{\bar{u}(\bar{x})}$ corresponding to the element $\bar{u}(\bar{x})$ of [1] by putting $f = \chi_{\bar{S}(\bar{x})}$, which is the characteristic fuction of $\bar{S}(\bar{x})$ defined in §2. According to [7] proposition (3.0), the integrals on the right hand side of (2.0) all vanish in the case of

$f = \chi_{\bar{S}(\bar{x})}$ with $\bar{x} = (\bar{\alpha}, \bar{\gamma})$ and $-\frac{\bar{\alpha}}{\bar{\gamma}} \notin (F^\times)^2$ except for that corresponding to $\bar{u}(\bar{x})$.

This makes us compute some subregular germs easily, which was done in [7]. But it is not so easy as expected to do for the pairs with $-\frac{\bar{\alpha}}{\bar{\gamma}} \in (F^\times)^2$ since at least two terms on the right hand side appear.

3. Computation of Jacobians.

Suppose that t is a regular element of T sufficiently close to identity and $t = x + id$. In case that $s \in \bar{S}_3(\bar{\alpha}, \bar{\gamma})$ is any element with $z_{12} \neq 0$ and with $(a-\alpha)^2 - \frac{\bar{\gamma}}{\bar{\alpha}} z_{12}^2$ squares, there exists $g \in G$ satisfying $t^g = s \in \bar{S}(\bar{x})$ for $\bar{x} = (\bar{\alpha}, \bar{\gamma})$ if and only if $\frac{2b\bar{\alpha}}{\bar{P}(\alpha - a)} \in N_F^{E^{\theta_1}}((E^{\theta_1})^\times)$ and $\frac{2\beta\bar{\alpha}}{\bar{Q}(a-\alpha)} \in N_F^{E^{\theta_2}}((E^{\theta_2})^\times)$, where

$$\bar{P} = 2 - 2a + z_{22} = \alpha - a \pm \sqrt{(\alpha - a)^2 - \frac{\bar{\gamma}}{\bar{\alpha}} z_{12}^2} \qquad \text{and}$$

$$\bar{Q} = 2 - 2\alpha + z_{22} = a - \alpha \pm \sqrt{(\alpha - a)^2 - \frac{\bar{\gamma}}{\bar{\alpha}} z_{12}^2}.$$

We constructed a composite mapping in [7] :

$$(T\backslash G \supset) \bar{G}(t) \xrightarrow{c^t} \bar{S}(\bar{x}) \xrightarrow{\hat{P}} \bar{S}_3(\bar{\alpha}, \bar{\gamma}) \times P^7 \xrightarrow{P'} P \times P^7$$

which was bijective except at $z_{12} = 0$ and at z_{12} in the form (1.3) which does not make $(a - \alpha)^2 - \frac{\bar{\gamma}}{\bar{\alpha}} z_{12}^2$ squares. The modulus of this composite map's Jacobian was just $|D(t)|^{\frac{1}{2}}$.

4. Computation of Orbital Integrals.

Now taking measures and notations as in [7] §5, we had

PROPOSITION 4.0.

(i) *If* $\frac{2b\bar{\alpha}}{\alpha - a} \in N_F^{E^{\theta_1}}((E^{\theta_1})^\times)$ *and* $\frac{2\beta\bar{\alpha}}{a - \alpha} \in N_F^{E^{\theta_2}}((E^{\theta_2})^\times)$, *then*

$$\int_{T\backslash G} \chi_{\bar{S}(\bar{\alpha}, \bar{\gamma})}(t^g) d\dot{g} = \bar{m}((X \cap Y) \cup (X' \cap Y')) \times q^{-7} \times |D(t)|^{-\frac{1}{2}}.$$

(ii) *If* $\frac{2b\bar{\alpha}}{\alpha - a} \in N_F^{E^{\theta_1}}((E^{\theta_1})^\times)$ *and* $\frac{2\beta\bar{\alpha}}{a - \alpha} \notin N_F^{E^{\theta_2}}((E^{\theta_2})^\times)$, *then*

$$\int_{T\backslash G} \chi_{\bar{S}(\bar{\alpha}, \bar{\gamma})}(t^g) d\dot{g} = \bar{m}((X \cap \bar{Y}) \cup (X' \cap \bar{Y}')) \times q^{-7} \times |D(t)|^{-\frac{1}{2}}.$$

(iii) *If* $\frac{2b\bar{\alpha}}{\alpha-a} \notin N_F^{E^{\theta_1}}((E^{\theta_1})^\times)$ *and* $\frac{2\beta\bar{\alpha}}{a-\alpha} \in N_F^{E^{\theta_2}}((E^{\theta_2})^\times)$, *then*

$$\int_{T\backslash G} \chi_{\bar{S}(\bar{\alpha},\bar{\gamma})}(t^g)d\dot{g} = \bar{m}((\bar{X} \cap Y) \cup (\bar{X}' \cap Y')) \times q^{-7} \times |D(t)|^{-\frac{1}{2}}.$$

(iv) *If* $\frac{2b\bar{\alpha}}{\alpha-a} \notin N_F^{E^{\theta_1}}((E^{\theta_1})^\times)$ *and* $\frac{2\beta\bar{\alpha}}{a-\alpha} \notin N_F^{E^{\theta_2}}((E^{\theta_2})^\times)$, *then*

$$\int_{T\backslash G} \chi_{\bar{S}(\bar{\alpha},\bar{\gamma})}(t^g)d\dot{g} = \bar{m}((\bar{X} \cap \bar{Y}) \cup (\bar{X}' \cap \bar{Y}')) \times q^{-7} \times |D(t)|^{-\frac{1}{2}}.$$

Now we have to look for the orbital integral over the conjugacy class of $\bar{u}(\bar{\alpha},\bar{\gamma})$. Also referring to [7] §5, we know that

$$\int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\backslash G} \chi_{\bar{S}(\bar{\alpha},\bar{\gamma})}(\bar{u}(\bar{\alpha},\bar{\gamma})^g)d\dot{g} = q^{-7}.$$

Moreover, we got from this

THEOREM 4.1. *In the case of* $-\frac{\bar{\alpha}}{\bar{\gamma}} \notin (F^\times)^2$, *the Shalika's unipotent subregular germs are obtained case by case as follows :*

(i) *If* $\frac{2b\bar{\alpha}}{\alpha-a} \in N_F^{E^{\theta_1}}((E^{\theta_1})^\times)$ *and* $\frac{2\beta\bar{\alpha}}{a-\alpha} \in N_F^{E^{\theta_2}}((E^{\theta_2})^\times)$, *then*

$$\Gamma_{\bar{u}(\bar{\alpha},\bar{\gamma})} = \bar{m}((X \cap Y) \cup (X' \cap Y')) \times |D(t)|^{-\frac{1}{2}}.$$

(ii) *If* $\frac{2b\bar{\alpha}}{\alpha-a} \in N_F^{E^{\theta_1}}((E^{\theta_1})^\times)$ *and* $\frac{2\beta\bar{\alpha}}{a-\alpha} \notin N_F^{E^{\theta_2}}((E^{\theta_2})^\times)$, *then*

$$\Gamma_{\bar{u}(\bar{\alpha},\bar{\gamma})} = \bar{m}((X \cap \bar{Y}) \cup (X' \cap \bar{Y}')) \times |D(t)|^{-\frac{1}{2}}.$$

(iii) *If* $\frac{2b\bar{\alpha}}{\alpha-a} \notin N_F^{E^{\theta_1}}((E^{\theta_1})^\times)$ *and* $\frac{2\beta\bar{\alpha}}{a-\alpha} \in N_F^{E^{\theta_2}}((E^{\theta_2})^\times)$, *then*

$$\Gamma_{\bar{u}(\bar{\alpha},\bar{\gamma})} = \bar{m}((\bar{X} \cap Y) \cup (\bar{X}' \cap Y')) \times |D(t)|^{-\frac{1}{2}}.$$

(iv) *If* $\frac{2b\bar{\alpha}}{\alpha-a} \notin N_F^{E^{\theta_1}}((E^{\theta_1})^\times)$ *and* $\frac{2\beta\bar{\alpha}}{a-\alpha} \notin N_F^{E^{\theta_2}}((E^{\theta_2})^\times)$, *then*

$$\Gamma_{\bar{u}(\bar{\alpha},\bar{\gamma})} = \bar{m}((\bar{X} \cap \bar{Y}) \cup (\bar{X}' \cap \bar{Y}')) \times |D(t)|^{-\frac{1}{2}}.$$

5. Subregular germs.

Let $-\frac{\bar{\alpha}}{\bar{\gamma}} \notin (F^{\times})^2$ and put $W(\bar{a},\bar{\alpha}) = \{\, z_{11} \in P : \sqrt{-\frac{\bar{\alpha}}{\bar{a}} z_{11}} \in F \,\}$. Then it is not difficult to know that

$$\int_{Z(u_{\bar{a}})\backslash G} \chi_{\bar{S}(\bar{u}(\bar{\alpha},\bar{\gamma}))}(u_{\bar{a}}^{g})\,d\dot{g} = \bar{m}(W(\bar{a},\bar{\alpha})) \times q^{-7},$$

where $u_{\bar{a}}$ are representatives of the form (1.2). By the way

$$\begin{aligned}\int_{T\backslash G} f(g^{-1}tg)\,d\dot{g} &= \Gamma_{\bar{u}(\bar{\alpha},\bar{\gamma})} \int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\backslash G} f(g^{-1}\bar{u}(\bar{\alpha},\bar{\gamma})g)\,d\dot{g} \\ &\qquad + \sum_{\bar{a}\in F^{\times}/(F^{\times})^2} \Gamma_{u_{\bar{a}}} \int_{Z(u_{\bar{a}})\backslash G} f(g^{-1}(u_{\bar{a}}g)\,d\dot{g} \\ &= q^{-7}\cdot\Gamma_{\bar{u}(\bar{\alpha},\bar{\gamma})} + \sum_{\bar{a}} \bar{m}(W(\bar{a},\bar{\alpha})) \times |D(t)|^{-\frac{1}{2}} \times q^{-7},\end{aligned}$$

where $f = \chi_{\bar{S}(\bar{u}(\bar{\alpha},\bar{\gamma}))}$. So, if we consider proposition (4.0), we have immediately

THEOREM 5.0. *In the case of* $-\frac{\bar{\alpha}}{\bar{\gamma}} \in (F^{\times})^2$, *the Shalika's unipotent subregular germs associated with a particular elliptic torus as in* [2] *hereof are obtained case by case as follows :*

(i) *If* $\frac{2b\bar{\alpha}}{\alpha-a} \in N_F^{E^{\theta_1}}((E^{\theta_1})^{\times})$ *and* $\frac{2\beta\bar{\alpha}}{a-\alpha} \in N_F^{E^{\theta_2}}((E^{\theta_2})^{\times})$, *then*

$$\Gamma_{\bar{u}(\bar{\alpha},\bar{\gamma})} = \{\bar{m}((X\cap Y)\cup(X'\cap Y')) - \sum_{\bar{a}} \bar{m}(W(\bar{a},\bar{\alpha}))\} \times |D(t)|^{-\frac{1}{2}}.$$

(ii) *If* $\frac{2b\bar{\alpha}}{\alpha-a} \in N_F^{E^{\theta_1}}((E^{\theta_1})^{\times})$ *and* $\frac{2\beta\bar{\alpha}}{a-\alpha} \notin N_F^{E^{\theta_2}}((E^{\theta_2})^{\times})$, *then*

$$\Gamma_{\bar{u}(\bar{\alpha},\bar{\gamma})} = \{\bar{m}((X\cap \bar{Y})\cup(X'\cap \bar{Y}')) - \sum_{\bar{a}} \bar{m}(W(\bar{a},\bar{\alpha}))\} \times |D(t)|^{-\frac{1}{2}}.$$

(iii) *If* $\frac{2b\bar{\alpha}}{\alpha-a} \notin N_F^{E^{\theta_1}}((E^{\theta_1})^{\times})$ and $\frac{2\beta\bar{\alpha}}{a-\alpha} \in N_F^{E^{\theta_2}}((E^{\theta_2})^{\times})$, then

$$\Gamma_{\bar{u}(\bar{\alpha},\bar{\gamma})} = \{\bar{m}((\bar{X}\cap Y)\cup(\bar{X}'\cap Y')) - \sum_{\bar{a}} \bar{m}(W(\bar{a},\bar{\alpha}))\} \times |D(t)|^{-\frac{1}{2}}.$$

(iv) *If* $\frac{2b\bar{\alpha}}{\alpha-a} \notin N_F^{E^{\theta_1}}((E^{\theta_1})^{\times})$ and $\frac{2\beta\bar{\alpha}}{a-\alpha} \notin N_F^{E^{\theta_2}}((E^{\theta_2})^{\times})$, then

$$\Gamma_{\bar{u}(\bar{\alpha},\bar{\gamma})} = \{\bar{m}((\bar{X}\cap \bar{Y})\cup(\bar{X}'\cap \bar{Y}')) - \sum_{\bar{a}} \bar{m}(W(\bar{a},\bar{\alpha}))\} \times |D(t)|^{-\frac{1}{2}}.$$

REMARK.

(A) *Note that the form (1.3) is never conjugate* $\bar{u}(\bar{\alpha}', \bar{\gamma}')$ *unless* $z_{12} = 0$ *in the case of* $\sqrt{-\frac{\bar{\alpha}}{\bar{\gamma}}} \in F^{\times}$.

(B) *It might not be conjectured easily that the subregular germs associated with any maximal torus in* $Sp_4(F)$ *should be the same as those in Theorem (5.0), although all Cartan subgroups are conjugate by some elements in* $Sp_4(\bar{F})$.

References

1. Artin E., *Algebraic Numbers and Algebraic Functions*, Gordon and Breach, 1967.
2. Asmuth C., *Some supercuspidal representations of* $Sp_4(k)$, Can. J. Math. **XXXIX** (1987), 1–7.
3. Borel A., *Linear Algebraic Groups*, W. A. Benjamin, 1969.
4. Jacobson N., *Basic Algebra I, II*, Freeman, 1980.
5. Kim Y., *Regular germs for p-adic* $Sp(4)$, Can. J. Math. **XLI** (1989), 626–641.
6. ———, *On whole regular germs for p-adic* $Sp_4(F)$, J. of Korean Math. Soc. **28** (1991), 207–213.
7. ———, *Some subregular germs for p-adic* $Sp_4(F)$, to appear.
8. Langlands R. and Shelstad D., *On the definition of transfer factors*, Preprint (1986).
9. Lang S., *Algebraic Number theory*, Springer–Verlag, 1986.
10. ———, $SL_2(\mathbf{R})$, Springer–Verlag, 1985.
11. Repka J., *Shalika's germs for p-adic* $GL(n)$ *I,II*, Pacific J. of Math. (1983).
12. ———, *Germs associated to regular unipotent classes in p-adic* $SL(n)$, Canad. Math. Bull. **28(3)** (1985).
13. Rogawski J., *An application of the building to orbital integrals*, Compositio Math. **42** (1981), 417–423.
14. Serre J., *Local Fields*, Springer–Verlag, 1979.
15. Shalika J. A., *A theorem on semi-simple p-adic groups*, Annals of Math. **95** (1972), 226–242.
16. Shelstad D., *A formula for regular unipotent germs*, Société Mathematique de France, Asterisque **171–172** (1989), 275–277.
17. Silberger A., *Introduction to Harmonic Analysis on Reductive p-adic Groups*, Princeton Univ., 1979.