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MINIMAL GENERIC SUBMANIFOLDS OF S^{2m+1} WITH FLAT NORMAL CONNECTION

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0. Introduction

Let M be an $(n+1)$-dimensional submanifold of a unit sphere S^{2m+1} of dimension 2m+1 with Sasakian structure tensors (ϕ, ξ, η, g). We suppose that M is tangent to the structure vector field ξ of S^{2m+1}. For any vector field X tangent to M, we put $\phi X = PX + FX$, where PX is the tangential part and FX the normal part of ϕX. If $\phi T_x(M)^\perp$ is contained in $T_x(M)$ for any point x of M, then M is called a generic submanifold of S^{2m+1}.

We define the notion of η-parallel second fundamental form of M. If the second fundamental form A of M satisfies the identity $g((\nabla_X A)_V Y, Z) = 0$ for any vector field X, Y and Z orthogonal to ξ and for any vector field V normal to M, then M is said to be η - parallel.

If the second fundamental form A of M satisfies $A_a P = PA_a$ for any direction V_a,$\{V_a\}$ being an orthonormal frame of the normal space, and if the normal connecton of M is flat , then the second fundamental form A of M is parallel (see [5]). On the other hand, we can see that $A_a P = PA_a$ for any direction V_a is equivalent to $(\nabla_\xi A)_a = 0$ for any direction V_a under the condition that the normal connection of M is flat.

The purpose of the present paper is to prove that if the second fundamental form A of a compact minimal generic submanifold with flat normal connection of S^{2m+1} is η-parallel, then A is parallel.

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1. Preliminaries

Let S^{2m+1} be a $(2m+1)$-dimensional unit sphere with Sasakian structure tensors (ϕ, ξ, η, g). The structure tensors of S^{2m+1} satisfy

$$\phi^2 X = -X + \eta(X)\xi, \qquad \phi\xi = 0, \qquad \eta(\xi) = 1, \qquad \eta(\phi X) = 0,$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad \eta(X) = g(X, \xi)$$

for any vector fields X and Y on S^{2m+1}. We denote by $\tilde{\nabla}$ the operator of covariant differentiation with respect to the metric tensor g on S^{2m+1}. We then have

$$\tilde{\nabla}_X \xi = \phi X, \qquad (\tilde{\nabla}_X \phi)Y = -g(X,Y)\xi + \eta(Y)X = \tilde{R}(X,\xi)Y,$$

$\tilde{R}$ denoting the Riemannian curvature tensor of S^{2m+1}. Let M be an $(n+1)$-dimensional submanifold of S^{2m+1}. Throughout this paper, we assume that the submanifold M of S^{2m+1} is tangent to the structure vector field ξ.

We denote by the same g the Riemannian metric tensor field induced on M from that of S^{2m+1}. The operator of covariant differentiation with respect to the induced connection on M will be denoted by ∇. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X,Y), \qquad \tilde{\nabla}_V X = -A_V X + D_X V$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the operator of covariant differentiaton with respect to the linear connection induced in the normal bundle $T(M)^{\perp}$ of M. A and B appearing here are both called the second fundamental forms of M and are related by

$$g(B(X,Y),V) = g(A_V X, Y).$$

The second fundamental form A_V in the direction of the normal vector V can be considered as a symmetric $(n+1, n+1)$-matrix.

The covariant derivative $(\nabla_X A)_V$ of A is defined to be

$$(\nabla_X A)_V Y = \nabla_X (A_V Y) - A_{D_X V} Y - A_V \nabla_X Y.$$

If $(\nabla_X A)_V Y = 0$ for any vector fields X and Y tangent to M, then the second fundamental form of M is said to be parallel in the direction of V.

If the second fundamental form is parallel in any direction, it is said to be parallel.

The mean curvature vector ν of M is defined to be $\nu = TrB/(n+1)$, where TrB denoting the trace of B. If $\nu = 0$, then M is said to be minimal. If the second fundamental form A vanishes identically, then M is said to be totally geodesic. A vector field V normal to M is said to be parallel if $D_X V = 0$ for any vector field X tangent to M.

For any vector field X tangent to M, we put

$$\phi X = PX + FX,$$

where PX is the tangential part and FX the normal part of ϕX.Then P is an endomorphism on the tangent bundle $T(M)$ and F is a normal bundle valued 1-form on the tangent bundle $T(M)$.

If $\phi T_x(M)^\perp$ is contained in $T_x(M)$ for any point x of M, then M is called a generic submanifold of S^{2m+1} (see [3]).

In the following we suppose that M is a generic submanifold of S^{2m+1}. Then, for any vector field V normal to M, ϕV is tangent to$M$. We also have

$$FP = 0, \qquad g(PX, Y) + g(X, PY) = 0,$$
$$g(FX, V) + g(X, \phi V) = 0.$$

For any vector field X tangent to M, we have

$$\tilde{\nabla}_X \xi = \phi X = \nabla_X \xi + B(X, \xi),$$

from which

$$\nabla_X \xi = PX, \qquad A_V \xi = -\phi V, \qquad B(X, \xi) = FX.$$

Furthermore, we see

$$(\nabla_X P)Y = A_{FY}X + \phi B(X, Y) - g(X, Y)\xi + \eta(Y)X,$$
$$(\nabla_X F)Y = -B(X, PY).$$
$$\nabla_X \phi V = -PA_V X + \phi D_X V.$$

We also have

$$A_{FX}Y - A_{FY}X = 0$$

for any vectors X and Y in $\phi T_x(M)^\perp$.

Moreover, equations of the Gauss and Codazzi of M are given respectively by

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + A_{B(Y,Z)}X - A_{B(X,Z)}Y,$$

where R being the Riemannian curvature tensor of M,

$$g((\nabla_X A)_V Y, Z) - g((\nabla_Y A)_V X, Z)$$

$$= g((\nabla_X B)(Y,Z),V) - g((\nabla_Y B)(X,Z),V) = 0.$$

We define the curvature tensor $R^{\perp}$ of the normal bundle of M by

$$R^{\perp}(X,Y)V = D_X D_Y V - D_Y D_X V - D_{[X,Y]}V.$$

Then we have equation of the Ricci

$$g(R^{\perp}(X,Y)V,U) + g([A_U, A_V]X,Y) = 0.$$

If $R^{\perp}$ vanishes identically, the normal connection of M is said to be flat.

2. The proof of Theorem

Let M be an $(n+1)$- dimensional generic submanifold of S^{2m+1}. If the second fundamental form A of M satisfies the identity $g((\nabla_X A)_V Y, Z) = 0$ for any vector fields X, Y and Z orthogonal to ξ and for any vector field V normal to M, then M is said to be η-parallel.

We prove the following

THEOREM 1. *Let M be a compact $(n+1)$-dimensional $(n > 4)$ minimal generic submanifold of S^{2m+1} with flat normal connection. If the second fundamental form A of M is η-parallel, then A is parallel.*

To prove the theorem we prepare some lemmas. We denote by S the Ricci tensor of M. Then we have generally (Yano [2])

$$div(\nabla_X X) - div((divX)X)$$

$$= S(X,X) + \frac{1}{2}|L(X)g|^2 - |\nabla X|^2 - (divX)^2,$$

where $(L(X)g)(Y,Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y)$ and $|\ \ |$ denotes the length of a tensor. If U is a parallel section in the normal bundle of M, then $\nabla_X \phi U = -PA_U X$. Hence we have $div\phi U = -TrPA_U = 0$ since P is skew symmetric and A_U is symmetric. Thus we also have $div((div\phi U)\phi U) = 0$. Substituting these equation into the equation above, we find

$$div(\nabla_{\phi U}\phi U) = S(\phi U, \phi U) + \frac{1}{2}\|[A_U, P]\|^2 + |\nabla\phi U|^2$$

by the equation $(L(tU)g)(X,Y) = g([A_U, P]X, Y)$. Since M is minimal, the Gauss equation implies

$$S(X,Y) = ng(X,Y) - \sum g(A_a X, A_a Y),$$

where A_a is the second fundamental form with respect to the direction $V_a, \{V_a\}$ being an orthonormal frame of the normal space.

On the other hand, we obtain

$$|\nabla\phi U|^2 = TrA_U{}^2 - g(\phi U, \phi U) - \sum g(A_a\phi U, A_a\phi U).$$

Combining the last three equations, we find

$$div(\nabla_{\phi U}\phi U) = (n+1)g(\phi U, \phi U) - TrA_U{}^2 + \frac{1}{2}\|[A_U, P]\|^2.$$

Therefore, we have

LEMMA 1. *Let M be an $(n+1)$-dimensional minimal generic submanifold of S^{2m+1} with flat normal connection. Then*

$$div(\sum \nabla_{\phi a}\phi V_a) = (n+1)P - \sum TrA_a{}^2 + \frac{1}{2}\sum \|[A_a, P]\|^2,$$

where $\nabla_{\phi a}$ is the covariant differentiation with respect to ϕV_a.

For any $(n+1)$-dimensional minimal submanifold of a unit sphere we have generally (Simons [1])

LEMMA 2. *Under the same assumptions like that of Lemma 1, we have*

$$-\frac{1}{2}\Delta|A|^2 + |\nabla A|^2 = \sum(TrA_aA_b)^2 - (n+1)|A|^2$$

Let $e_0 = \xi, e_1, ..., e_n$ be a local field of orthonormal frames of M. We use the convention that the ranges of indices are $t, s, r = 1, ..., n$. To simplify the notation, we put ∇_t the covariant differentiation with respect to e_t.

Since we have

$$(\nabla_t A)_a \xi = \nabla_t(A_a\xi) - A_a(\nabla_t \xi) = [P, A_a]e_t$$

for any a and t, it follows that

$$|\nabla A|^2 = g(\nabla A, \nabla A) = \sum g((\nabla_t A)_a e_s, e_r)^2 + 3\sum g((\nabla_t A)_a \xi, e_s)^2$$
$$= \sum g((\nabla_t A)_a e_s, e_r)^2 + 3\sum \|[A_a, P]\|^2.$$

From Lemma 2 we obtain

$$-\frac{1}{2}\Delta|A|^2 + |\nabla A|^2 \geq \sum(TrA_a^2)^2 - (n+1)\sum TrA_a^2.$$

Therefore we have the inequality

$$-\frac{1}{2}\Delta|A|^2 + \sum g((\nabla_t A)_a e_s, e_r)^2$$
$$\geq \sum(TrA_a{}^2)^2 - (n+1)\sum TrA_a{}^2 + 3\sum \|[A_a, P]\|^2.$$

Using Lemma 1, the right hand side of the inequality above reduces to

$$\sum(TrA_a{}^2)^2 - (n+1)\sum TrA_a{}^2 + 3\sum \|[A_a, P]\|^2$$
$$= \sum(TrA_a{}^2)^2 - (n+7)\sum TrA_a{}^2 + 6(n+1)p - 3div(\sum \nabla_{\phi a}\phi V_a)$$
$$= \sum\{TrA_a{}^2 - 6\}\{TrA_a{}^2 - (n+1)\} - 6div(\sum \nabla_{\phi a}\phi V_a)$$
$$= \sum\{TrA_a{}^2 - (n+1)\}^2 + (n-5)\{\sum TrA_a{}^2 - (n+1)p\} - div(\sum \nabla_{\phi a}\phi V_a).$$

Consequentry, we obtain

THEOREM 2. *Let M be a compact $(n+1)$-dimensional minimal generic submanifold of S^{2m+1} with flat normal connection. Then*

$$\int_M \sum g((\nabla_t A)_a e_s, e_r)^2 * 1 \geq \frac{1}{2}(n-5)$$
$$\int_M \sum \|[A_a, P]\|^2 * 1 + \int_M \sum\{TrA_a{}^2 - (n+1)\}^2 * 1.$$

From Theorem 2, if $n > 4$ and if the second fundamental form A of M is η-parallel, then $TrA_a{}^2 = n+1$ for all a, and hence $\sum TrA_a{}^2 = (n+1)p$. Then, by Lemma 1, we see $[A_a, P] = 0$, i.e., $A_aP = PA_a$ for all a. Moreover, we see that $(\nabla_t A)_a\xi = 0$ for all t and a. Consequently, the second fundamental form A of M is parallel. From these considerations and Theorems in [4] we have

THEOREM 3. *Let M be a compact $(n+1)$-dimensional $(n > 4)$ minimal generic submanifold with flat normal connection of S^{2m+1}. If the second fundamental form of M is η-parallel, then M is*

$$S^{m(1)}(r_1) \times \cdots \times S^{m(k)}(r_k),$$

$$r_t = (m(t)/(n+1))^{1/2}(t = 1, ..., k), \quad n+1 = \sum m(t),$$

where $m(1), ..., m(k)$ are odd numbers such that $0 < m(1), ..., m(k) < n+1$, codimension $p = k-1$.

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