Honam Mathematical J. 15(1993), No. 1,pp. 11–16

A MONOTONICITY INEQUALITY FOR UNIT VECTOR FIELDS

GIE-HYUN PARK

Dept. of Mathematics, Cheju National University, Cheju 690-756, Korea.

0. Introduction

In this paper we prove a monotonicity inequality and apply it to study regularity for energy minimizing unit vector fields on a Riemannian manifold. This study is motivated by the liquid crystal theory, where liquid crystals can be regarded as a vector field with fixed length.

If the tangent bundle is trivial like orientable compact 3-folds, unit vector fields may be regarded as a map to the sphere. This shows a strong relationship between the study of energy minimizing unit vector fields and that of energy minimizing maps into the sphere. Schoen and Uhlenbeck developed a regularity theory for minimizing harmonic maps into Riemannian manifolds, where they used the monotonicity inequality as a starting point [5]. Thus proving the monotonicity inequality leads finally to a regularity theory via Morrey's Lemma with some efforts.

Let M be an n-dimensional Riemannian manifold with metric g. We may assume that M is isometrically embedded in Euclidean space $\mathbb{R}^k$. We define our function space as

$$H_1^2(M,1) = \{u \in H_1^2(M,\mathbb{R}^k) : |u(x)| = 1 \text{ a.e. } x \in M\},$$

where $H_1^2(M,\mathbb{R}^k)$ is the Sobolev space of functions f with $|\nabla f|$ in L^2.

Let $u \in H_1^2(M,1)$ be a unit vector field on M. The energy $W_U(u,g)$ of u on $U \subset M$ is defined as

$$W_U(u,g) = \frac{1}{2}\int_U |\nabla u|^2,$$

where the connection is induced from g.

Received Octorber 13, 1992.

This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1991.

A unit vector field u is called energy minimizing on $U \subset M$ if for any unit vector v with $u|_{\partial U} = v|_{\partial U}$ the energy $W_U(u,g)$ is smaller than or equal to $W_U(v,g)$. A unit vector field u is called an energy minimizer on the manifold M if u is energy minimizing on any precompact subset $U \subset M$.

1. The Euler-Lagrange equation and monotonicity inequality

We compute the first variation of W, for which we take $u_t(x) = (u(x)+t\phi(x))/|u(x)+t\phi(x)|$ as a one-parameter variation through u, where $\phi \in H_1^2(M, \mathbb{R}^k)$ is compactly supported, and obtain;

LEMMA 1. *If u is energy minimizing on M and $|u| = 1$ a.e., the u satisfies the following equation in weak sense;*

$$\Delta u + |\nabla u|^2 u = 0 \tag{1}$$

For a point $p \in M$, let $B_M(p,\rho) \in M$ be a ball of radius ρ centered at p and let $Exp_p : T_pM \to M$ be the exponential map. We can take uniformly ρ which is so small that Exp_p is a diffeomorphism from $B_{T_pM}(0,\rho)$ to $B_M(p,\rho)$. Denote the set $B_{T_pM}(0,\rho)$ by $B(0,\rho) = B_\rho$ and pull back the metric from M using Exp_p. On B_ρ this metric is of the form $ds^2 = dr^2 + dA(r)^2$ where $dA(r)^2$ is the metric restricted to the sphere of radius r. We may assume that u is defined on B_ρ. We consider the blow-up sequence of vector fields u_ρ on B_1 with norm 1 with respect to the pull-back metric. More precisely for $\phi_\rho(\cdot) = Exp_p(\rho\cdot) : B_1 \to B_M(p,\rho) \in M$ let $u_\rho = d{\phi_\rho}^{-1}(u)$, which is an energy minimizer with respect to the pull-back metric if u is one on M.

We denote the energy of a unit vector field u on a set U with respect to the Euclidean metric as $E_U(u)$.

On B_1 the pull-back metric $ds^2 = (g_{\alpha\beta})$ satisfies $g_{\alpha\beta} = \delta_{\alpha\beta}$ and for some $\Lambda > 0$

$$\sum_{\alpha,\beta,\tau} \left| \frac{\partial}{\partial x^\tau} g_{\alpha\beta} \right| \leq \Lambda. \tag{2}$$

We denote $\mathcal{G}_\Lambda$ the set of all Riemannian metrics on B_1 satisfying (2) and of the form $ds^2 = dr^2 + dA(r)^2$.

We use this inequality to estimate $W_{B_\rho}(v)$ by $E_{B_\rho}(v)$ and vice versa.

Let W_ρ and E_ρ denote W_{B_ρ} and E_{B_ρ}. We have

$$\begin{aligned}|E_\rho(u) - W_\rho(u)| \leq & c\Lambda(\rho E_\rho(u) + \rho^{n/2}E_\rho(u)^{1/2} + \Lambda\rho^n) \\ \leq & \frac{3}{2}c\Lambda(\rho E_\rho(u) + \rho^{n-1})\end{aligned}$$

provided $\Lambda\rho \leq 1$. Consequently for $\rho \in (0,1]$ we have

(3) $\quad (1 - c\Lambda\rho)E_\rho(u) - c\Lambda\rho^{n-1} \leq W_\rho(u) \leq (1 + c\Lambda\rho)E_\rho(u) + c\Lambda\rho^{n-1}$

provided $c\Lambda \leq \frac{1}{2}$. This shows that the topology of $H_1^2(B_\rho, N)$ is well defined regardless of the metric on B_ρ provided Λ is sufficiently small.

We now prove:

PROPOSITION 2. (Monotonicity inequality) *If u is energy minimizing, then we have*

$$\sigma^{2-n}W_\sigma(u) \leq \rho^{2-n}W_\rho(u).$$

for $\sigma \leq \rho \leq 1$.

Proof. For almost all $\sigma \in (0,1]$ we have $\int_{|x|=\sigma}|\nabla u|^2 d\xi < \infty$ where ξ is a variable on the sphere. Introduce the comparison map

$$v_\sigma(x) = \begin{cases} u(x), & |x| \geq \sigma \\ \text{parallel translate of } u(\frac{\sigma x}{|x|}) \text{ through raidal direction,} & |x| \leq \sigma. \end{cases}$$

Since the result is trivial for $n = 2$, we assume $n > 2$. Denote by $|\nabla_\xi u|^2$ the tangential energy along the spheres $|x| = r$, so that $|\nabla u|^2 = |\nabla_\xi u|^2 + |\partial u/\partial r|^2$ holds since $g \in \mathcal{G}_\Lambda$. We compute

$$\begin{aligned}& W_\sigma(u) \\ \leq & W_\sigma(v_\sigma) \\ = & (n-2)^{-1}\sigma\int_{|x|=\sigma}|\nabla_\xi u|^2 d\xi \\ = & (n-2)^{-1}\sigma\left(\frac{d}{d\sigma}W_\sigma(u) - \int_{|x|=\sigma}|\frac{\partial u}{\partial r}|^2 d\xi\right).\end{aligned}$$

This implies

(4) $$0 \leq \sigma^{2-n}\int_{|x|=\sigma}|\frac{\partial u}{\partial r}|^2 d\xi \leq \frac{d}{d\sigma}\left[\sigma^{2-n}W_\sigma(u)\right]$$

Since $W_\sigma(u)$ is a nondecreasing function, we can integrate this inequality from σ to ρ.

$$\sigma^{2-n}W_\sigma(u) \leq \rho^{2-n}W_\rho(u)$$

2. Partial regularity

Using this monotonicity inequality we can prove that a blow-up of an energy minimizer converges. Let $\rho_\lambda(x) = \lambda x : B_1 \to B_\lambda$. We set $u_\lambda = u \circ \rho_\lambda$ for $\lambda \in (0,1]$. Then if u is $W(\ ,g)$-minimizing, so is u_λ for the metric g_λ induced by the map ρ_λ, and

$$W_1(u_\lambda, g_\lambda) = \lambda^{2-n} W_\lambda(u,g).$$

This and the monotonicity inequality show that $W_1(u_\lambda, g_\lambda)$ is uniformly bounded for $\lambda \le 1$, so is $E_1(u)$ by (2) and therefore there is a convergent subsequence with limit $u_0 \in H_1^2(B_1, \mathbb{R}^n)$. Furthermore we have:

LEMMA 3. *We can find a sequence $\lambda(k) \in (0,1]$ converging to 0, such that $u_\lambda(k)$ converges weakly in $H_1^2(B_1, 1)$ to a map $u_0 \in H_1^2(B_1, S^{n-1})$ with $\partial u_0/\partial r = 0$ a.e. in B_1. Furthermore u_0 is energy minimizing as a map from B_1 to sphere in the Euclidean metric.*

Proof. We integrate (4) from 0 to λ with respect to the radial direction to get

$$\int_{B_\lambda} r^{2-n} \left| \frac{\partial u}{\partial r} \right| \le \lambda^{2-n} W_\lambda(u) - \lim_{\sigma \to 0} \sigma^{2-n} W_\sigma(u)$$

Thus by a change of variables we have

$$\lim_{\lambda \to 0} \int_{B_1} r^{2-n} \left| \frac{\partial u_\lambda}{\partial r} \right| = \lim_{\lambda \to 0} \int_{B_\lambda} r^{2-n} \left| \frac{\partial u}{\partial r} \right| = 0.$$

This shows that $\partial u_0/\partial r = 0$ a.e.

The last statement follows from Proposition 5.1 in [3] and the fact that the metric induced by the map ρ_λ converges to the Euclidean one as λ goes to 0.

Here we state the regularity theorem due to Schoen and Uhlenbeck. Let $u : B_1^n \to N$ be an energy minimizing map into a Riemannian manifolds with its image in a compact set $N_0 \subset N$ with respect to a metric $g \in \mathcal{G}_\Lambda$.

REGULARITY ESTIMATE. [5] *There exists $\varepsilon > 0$ depending only on n and N_0 such that if u is energy minimizing, and $\Lambda < \varepsilon$, then u is Holder continuous on $B_{1/2}$ and satisfies $|u(x) - u(y)| \le c|x-y|^\alpha$ for $x, y \in B_{1/2}$ where $c, \alpha > 0$ depends only on n, N_0.*

To prove this for our situation we need some modification of the original proof such as the projection Π from neighborhood of N to N replaced by $\Pi : v \to v/|v|$, since in our case there is no fixed manifold N.

From this we can infer by using the covering argument that the $n-2$-dimensional Hausdorff measure of singular set is zero and that the singular set is discrete if $n=3$ [5].

Furthermore adapting the original proof by Liao [4] we can prove the following;

REGULARITY OF MAPS WITH SMALL ENERGY. [4] *Let u be an energy minimizing unit vector field which is smooth on $B-0$. There is a constant $\varepsilon>0$ independent of u such that 0 is a removable singularity if $E(u)\leq\varepsilon$.*

3. Two and three dimensional cases

For some 2-dimensional Riemannian manifolds, it makes no sense saying about energy minimizing unit vector field.

PROPOSITION 4. *On an oriented compact surface with non-zero genus, every unit vector field has infinite energy.*

Proof. If it has finite energy, it must be smooth. But topological constraint prohibits it.

It is well known that the tangent bundles of 3-dimensional oriented compact manifolds are trivial. So we may ask a question.

QUESTION. *On 3-dimensional oriented compact manifolds, are there smooth energy minimizing unit vector fields?*

Of cause, if a 3-fold has a boundary and if we ask whether there is a smooth energy minimizing vector field satisfying given boundary condition, the answer is negative [1].

PROPOSITION 5. *Let g_0 be the metric on S^3 induced from the Euclidean one of $\mathbb{R}^4$. There is a constant c such that if a metric g on S^3 satisfies $\|g-g_0\|_1\leq c$ then there is a smooth energy minimizing unit vector field on (S^3,g).*

Proof. (S^3,g_0) has a metric compatible Lie group structure so that at a point(identity as a point in a Lie group) a vector can be parallel-translated to any point in unique way giving a parallel smooth vector field v on S^3. Thus the energy of v is 0 in metric g_0.

Let g be a Riemannian metric on S^3. Since $W(\cdot,g)$ is coercive and weakly lower semi-continuous on $H_1^2\ ((S^3,g),1)\neq\emptyset$ with respect to $H_1^2\ ((S^3,g),\mathbb{R}^3)$ $W(\cdot,g)$ attains its minimum in $H_1^2\ ((S^3,g),1)$. Denote a minimizing unit vector field as w.

Then using (3) and $W(w) \leq W(v)$, it can be shown that $W(w)$ is small if $\|g - g_0\|_1 \leq c$ for small c. Therefore by regularity of maps with small energy w is smooth.

At the singular points of an energy minimizer u, using Lemma 3 we can show that the blow-up of u at the point is a minimizing tangent map to the sphere S^2 . Thus we have by [2];

PROPOSITION 6. *For an energy minimizing vector field on a 3-fold, at the singular points, its topological degree is* ± 1.

References

1 F. J., Almgren and E. H., Lieb, *Singularities of energy minimizing maps from the ball to the sphere: Examples, counterexamples, and bounds*, Ann. Math. **128** (1988), 483–530.

2 H. Brezis, J. M. Coron, and E. H. Lieb,, *Harmonic maps with defects*, Comm. Math. Phys. **107** (1986), 649–705.

3 R. Hart, D. Kinderlehrer, and F.-H. Lin, *Stable defects of minimizers of constrained variational principles*, Ann. Inst. Henri Poincare **5** (1988), 297–322.

4 G. Liao, *A regularity theorem for harmonic maps with small energy*, J. Diff. Geom. **22** (1985), 233–241.

5 R. Schoen and K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Diff. Geom. **17** (1982), 307–335.

6 R. Schoen and K. Uhlenbeck, *Regularity of minimizing harmonic maps into sphere*, Invent. Math. **78** (1984), 89–100.