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SLICE MAPS FOR THE CROSSED PRODUCTS OF C*–ALGEBRAS

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1. Introduction

Whenever we consider tensor products of *complete* objects such as Banach spaces, Banach algebras, C^*-algebras or von Neumann algebras, we should define a suitable norm on the algebraic tensor product and take the completion. Although there are so many possibilities to define norms to get the same objects, we usually have a typical norm under which the following property holds:

$$C \subseteq A,\ D \subseteq B \quad \Longrightarrow \quad C \otimes D \subseteq A \otimes B. \tag{1.1}$$

Even if we take the tensor product with the above property, there are so many pathological phenomena in view of algebraic tensor products. The question of exactness for C^*-algebras is one of the typical examples among them. It is well known that the minimal C^*-tensor product satisfies the condition (1.1) and we denote by $A \otimes B$ for the minimal tensor product of C^*-algebra A and B.

Wassermann [**19**, **20**] showed that the following sequence

$$0 \to B \otimes I \to B \otimes A \to B \otimes A/I \to 0 \tag{1.2}$$

need not to be exact. We say that a C^*-algebra B is *exact* if the sequence (1.2) is exact for any C^*-algebra A and its two-sided norm-closed ideal I. The class of exact C^*-algebras is very important among C^*-algebras which includes strictly nuclear C^*-algebras, and is stable under forming C^*-subalgebras, inductive limits, C^*-quotients [**1**, **10**, **12**]. There are so many equivalent conditions for C^*-exactness [**8**, **9**, **11**] and the notion

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of C^*-exactness plays a rôle in the theory of C^*-algebras independent of tensor products. For recent developments, we refer to [**2, 7, 21, 22**] for examples. Note that the maximal C^*-tensor product satisfies the condition (1.2) although it does not satisfy (1.1) [**6**].

It has been turned out that the notion of slice maps are very useful to deal with the sequence (1.2) [**8, 16, 19**]. For each $\phi \in B^*$, we can define a bounded linear map $R_\phi : B \otimes A \to A$, said to be the *right slice map* associated with ϕ, satisfying

$$R_\phi(b \otimes a) = \phi(b)a \tag{1.3}$$

for $b \in B$ and $a \in A$. The *Fubini product* $F(B, I)$ is defined by

$$F(B, I) = \{x \in B \otimes A : R_\phi(x) \in I \text{ for all } \phi \in B^*\}. \tag{1.4}$$

Then, it is easy to see that $F(B, I)$ is exactly the kernel of the $*$-homomorphism $B \otimes A \to B \otimes A/I$, and so the question of exactness of the sequence (1.2) is reduced to the question when the Fubini product $F(B, I)$ coincides with the ordinary minimal tensor product $B \otimes I$. The notions of slice maps and Fubini products may be defined for the spatial tensor products of σ-weak closed operator spaces or the Haagerup tensor products of operator spaces [**13, 15**].

In this note, we consider the same question of exactness for the crossed product of C^*-algebras. Let G be a locally compact group and α an action of G on a C^*-algebra A. If I is a norm-closed two-sided ideal of A which is invariant under the action α then α also acts on the quotient C^*-algebra A/I. Our question is whether the sequence

$$0 \to G \ltimes_{\alpha r} I \to G \ltimes_{\alpha r} A \to G \ltimes_{\alpha r} A/I \to 0 \tag{1.5}$$

is exact or not, where $G \ltimes_{\alpha r} A$ denotes the *reduced crossed product* of A and G. This question has been considered by Zeller-Meier [**23**] for discrete groups. If α is the trivial action, then our question becomes whether the reduced group C^*-algebra $C_r^*(G)$ is exact or not. It is well-known that if G is an amenable group or G is the free group on two generators then $C_r^*(G)$ is exact [**1**]. Very recently, the authors heard that several authors showed that reduced group C^*-algebras $C_r^*(G)$ are exact for various classes of discrete groups. Nevertheless, the authors do not know whether the sequence (1.5) is exact or not in general. We will define the slice map and Fubini product in this case, and examine the relation with the sequence (1.5).

2. Dual spaces of reduced group C*-algebras

In this section, we review the definition of crossed product of C^*-algebras together with the dual spaces of reduced group C^*-algebras to get a motivation. Let G be a locally compact group and α a continuous action of G on a C^*-algebra A, that is, a continuous homomorphism of G into the group $\mathrm{Aut}(A)$ of $*$-automorphism of A equipped with the topology of pointwise convergence. The set $K(G, A)$ of continuous functions from G into A with compact supports is a $*$-algebra under the operations

$$y^*(t) = \Delta(\cdot)^{-1}\alpha_t(y(t^{-1}))^*$$
$$(y \times z)(t) = \int_G y(s)\alpha_s(z(s^{-1}t))ds,$$

for $y, z \in K(G, A)$, where ds is the left-invariant Haar measure on G and Δ denotes the modular function of G. Denote by $L^1(G, A)$ the completion of $K(G, A)$ with respect to the norm

$$\|y\|_1 = \int_G \|y(s)\|ds$$

for $y \in K(G, A)$.

For a representation π of A on a Hilbert space H, we define the representation $\mathrm{Ind}\pi$ of $L^1(G, A)$ on $L^2(G, H)$ by

$$((\mathrm{Ind}\pi)(y)(\xi))(t) = \int_G \pi(\alpha_{t^{-1}}(y(s)))\xi(s^{-1}t)ds$$

for $y \in L^1(G, A)$ and $\xi \in L^2(G, H)$. Note that $\mathrm{Ind}\pi$ is associated with the covariant representation $(\tilde{\pi}, \lambda, L^2(G, H))$ given by

$$(\tilde{\pi}(x)\xi)(t) = \pi(\alpha_{t^{-1}}(x))\xi(t)$$
$$(\lambda(s)\xi)(t) = \xi(s^{-1}t)$$

for $x \in A, s \in G$ and $\xi \in L^2(G,H)$. The *reduced crossed product* of G by α, denoted by $G \ltimes_{\alpha r} A$, is the completion of $L^1(G,A)$ with respect to the norm

$$\|x\|_r = \sup\{\|(\mathrm{Ind}\pi)(x)\| : \pi \in \mathrm{Rep}A\},$$

where Rep A is the set of all representations of A. For a detailed discussion, we refer to [**14**, Chapter 7].

If α is the trivial action then $G \ltimes_{\alpha r} A$ is nothing but $C_r^*(G) \otimes A$, where $C_r^*(G)$, the *reduced group C^*-algebra* associated with G, is by definition just $G \ltimes_{\alpha r} \mathbb{C}$ with the trivial action α on the complex field $\mathbb{C}$. Hence, we should consider the dual space of $C_r^*(G)$ in order to modify (1.3). To do this, we summarize parts of [**3**, **4**, **5**]. Note that a unitary representation π of a locally compact group G corresponds to a non-degenerate representation π (we use the same notation) of the Banach $*$-algebra $L^1(G)$ by the relation

$$\pi(x) = \int_G x(s)\pi(s)ds \in B(H_\pi),$$

for $x \in L^1(G)$. The *full group C^*-algebra* $C^*(G)$ is defined by the completion of $L^1(G)$ with respect to the norm

$$\|x\|_c = \sup\{\|\pi(x)\| : \pi \text{ runs unitary representations of } G\}.$$

Note that the restriction map from $C^*(G)^*_+$ into $L^1(G)^*_+$ is one-to-one onto. It is even norm-preserving because we can choose an approximate identity $\{\mu_i\}$ both in $C^*(G)$ and $L^1(G)$ with $\|\mu_i\|_c = \|\mu_i\|_1 = 1$ [**5**, Lemma 1.4]. Now, if $\phi \in L^\infty(G)$ is considered as a positive functional on $L^1(G)$ then it is associated (in the sense of the relation (2.1) below) with a unitary representation π of G and a vector ξ in $L^2(G)$. Indeed, if (π, ξ) is the Gelfand-Naimark-Segal construction of $L^1(G)$ associated with ϕ, then we have

$$\int \phi(s)x(s)ds = \langle \phi, x\rangle = \langle \pi(x)\xi, \xi\rangle = \int x(s)\langle \pi(s)\xi, \xi\rangle ds,$$

for each $x \in L^1(G)$, and so, it follows that

$$\phi(s) = \langle \pi(s)\xi, \xi\rangle. \tag{2.1}$$

Such a function ϕ is said to be a *continuous positive definite* function on G, and they form the positive cone of $C^*(G)^*$, denoted by $P(G)$.

Now, for each $\xi \in L^2(G)$, we denote by ϕ_ξ the continuous positive definite function associated with the left regular representation λ and ξ as follows:

$$\phi_\xi(s) = \langle \lambda(s)\xi, \xi \rangle = \int \xi(s^{-1}t)\overline{\xi(t)}dt.$$

Then, the positive cone $P_r(G)$ of $C_r^*(G)^*$ is the set of all continuous positive definite functions which are *weakly associated* with ϕ_ξ's, or equivalently, the limits of sums of ϕ_ξ's, with respect to the compact-open topology.

3. Slice maps for crossed products

By the above discussion, we know that every bounded linear functional on $C_r^*(G)$ is essentially associated with a vector $\xi \in L^2(G)$, and so we associate a linear map $R_\xi : L^1(G, A) \to A$ for each $\xi \in L^2(G)$ as follows:

$$R_\xi(y) = \iint \xi(s^{-1}t)\overline{\xi(t)}\alpha_{t^{-1}}(y(s))dsdt, \qquad y \in L^1(G, A). \tag{3.1}$$

PROPOSITION 3.1. *The map (3.1) can be extented to a bounded linear map $R_\xi : G \ltimes_{\alpha r} A \to A$, with $\|R_\xi\| \le \|\xi\|_2^2$.*

Proof. It suffices to show that

$$\|R_\xi(x)\| \le \|\xi\|_2^2 \|x\|_r,$$

for $x \in L^1(G, A)$. Let π be a representation of A on a Hilbert space H, and η_1, η_2 be unit vectors in H. Then, we have

$$\begin{aligned}
&\langle \pi(R_\xi(x))\eta_1, \eta_2 \rangle \\
=&\iint \xi(s^{-1}t)\overline{\xi(t)}\langle \pi(\alpha_{t^{-1}}(x(s))\eta_1, \eta_2\rangle dsdt \\
=&\int \langle \int \pi(\alpha_{t^{-1}}(x(s))\xi(s^{-1}t)\eta_1 ds, \xi(t)\eta_2 \rangle dt \\
=&\int \langle (\mathrm{Ind}\pi)(x)(\xi \otimes \eta_1)(t), (\xi \otimes \eta_2)(t) \rangle dt \\
=&\langle (\mathrm{Ind}\pi)(x)(\xi \otimes \eta_1), \xi \otimes \eta_2 \rangle.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}|\langle\pi(R_\xi(x))\eta_1,\eta_2\rangle| &\le \|(\mathrm{Ind}\pi)(x)\|\|\xi\otimes\eta_1\|\|\xi\otimes\eta_2\|\\ &= \|\xi\|_2^2\|(\mathrm{Ind}\pi)(x)\| \le \|\xi\|_2^2\|x\|_r,\end{aligned}$$

and so, we have

$$\|R_\xi(x)\| \le \|\xi\|_2^2\|x\|_r.$$

This completes the proof with the relation $\|R_\xi\| \le \|\xi\|_2^2$.

For $\xi\in L^2(G)$ and $\eta\in H$, define positive linear functionals Ψ^π_η and $\Phi^\pi_{\xi,\eta}$ of A and $G\ltimes_{\alpha r}A$, respectively, by

$$\begin{aligned}&\langle a,\Psi^\pi_\eta\rangle = \langle\pi(a)\eta,\eta\rangle, \qquad a\in A\\ &\langle x,\Phi^\pi_{\xi,\eta}\rangle = \langle(\mathrm{Ind}\pi)(x)(\xi\otimes\eta),(\xi\otimes\eta)\rangle, \qquad x\in G\ltimes_{\alpha r}A.\end{aligned}$$

Then, we have

$$\begin{aligned}\langle R_\xi(x),\Psi^\pi_\eta\rangle &= \iint \xi(s^{-1}t)\overline{\xi(t)}\langle\pi(\alpha_{t^{-1}}(x(s)))\eta,\eta\rangle\, ds dt\\ &= \langle(\mathrm{Ind}\pi)(x)(\xi\otimes\eta),(\xi\otimes\eta)\rangle\\ &= \langle\Phi^\pi_{\xi,\eta},x\rangle\end{aligned}$$

for $x\in G\ltimes_{\alpha r}A$, by the same calculation as in the above proof of Proposition 3.1. Because $\{\Phi^\pi_{\xi,\eta} : \pi\in\mathrm{Rep}A,\ \xi\in L^2(G),\ \eta\in H\}$ is a separating subset of $(G\ltimes_{\alpha r}A)^*$, we have

(3.2) $\qquad x=0 \quad\Longleftrightarrow\quad R_\xi(x)=0 \text{ for every } \xi\in L^2(G).$

4. Fubini products for reduced crossed products

Now, we modify (1.4) to define the *Fubini product* $F(G,C)$ for α-invariant C^*-subalgebra C of A as follows:

(4.1) $\qquad F(G,C)=\{x\in G\ltimes_{\alpha r}A : R_\xi(x)\in C \text{ for all } \xi\in L^2(G)\}.$

If α is the trivial action, then $G \ltimes_{\alpha r} A$ is nothing but the minimal tensor product $C_r^*(G) \otimes A$ of $C_r^*(G)$ and A as mentioned before. In this case, we have

$$\begin{aligned} R_\xi(x) &= \iint \xi(s^{-1}t)\overline{\xi(t)}x(s)dsdt \\ &= \int \langle \lambda(s)\xi, \xi \rangle x(s)ds \\ &= \int \phi_\xi(s)x(s)ds, \end{aligned}$$

where $\phi_\xi(s) = \langle \lambda(s)\xi, \xi \rangle$ is the continuous positive definite function of G associated with the left regular representation λ of G and $\xi \in L^2(G)$, which is a positive linear functional of $C_r^*(G)$ with $\|\varphi_\xi\| = \|\xi\|^2$. Hence, R_ξ is just the left slice map $R_{\phi_\xi} : C_r^*(G) \otimes A \to A$ associated with φ_ξ. Therefore, $F(G, C)$ is just the Fubini product $F(C_r^*(G), C)$ of $C_r^*(G)$ and C, and so our definition (4.1) is compatible with (1.4).

If G is a discrete group, then there were already the notions of slice maps and Fubini products as follows: Zeller-Meier **[23]** showed that there exists a unique bounded linear map $R_s : G \ltimes_{\alpha r} A \to A$ such that

$$R_s(y) = y(s), \qquad y \in \ell^1(G, A) \tag{4.2}$$

for each $s \in G$ and the kernel of the $*$-homomorphism $G \ltimes_{\alpha r} A \to G \ltimes_{\alpha r} (A/I)$ coincides with the set

$$\{y \in G \ltimes_{\alpha r} A\,;\, R_s(y) \in I \text{ for all } s \in G\}. \tag{4.3}$$

There is an another approach to define the linear map (4.2) using a positive faithful norm one projection from $G \ltimes_{\alpha r} A$ onto A **[17, 18]**. Note that if $G = \mathbb{Z}$ and $A = \mathbb{C}$ then $\mathbb{Z} \ltimes_{\alpha r} \mathbb{C}$ is the C^*-algebra $C_r^*(\mathbb{Z}) = C(\mathbb{T})$ of all continuous functions on the one-dimensional torus and the set $\{R_n(f) : n \in \mathbb{Z}\}$ is nothing but the usual Fourier coefficients of the function $f \in C(\mathbb{T})$.

Now, for each $y \in \ell^1(G, A)$, we have

$$R_\xi(y) \in C \text{ for every } \xi \in \ell^2(G) \iff y(s) \in C \text{ for every } s \in G.$$

Indeed, if $\xi \in \ell^2(G)$ is defined by $\xi(e) = \lambda$, $\xi(a^{-1}) = \mu$ and $\xi(s) = 0$ for $s \neq e, s \neq a^{-1}$, then we have

$$R_\xi(y) = |\lambda|^2 y(e) + \overline{\lambda}\mu y(a) + \lambda\overline{\mu}\alpha_a(y(a^{-1})) + |\mu|^2 \alpha_a(y(e)).$$

By choosing suitable λ and μ, we can express $y(a)$ as the linear combination of $R_\xi(y)$'s. Hence, our definition of Fubini products (4.1) is also compatible with (4.3).

PROPOSITION 4.1. *If I is an α-invariant closed two-sided ideal of A then the kernel of the $*$-homomorphism $G \ltimes_{\alpha r} A \to G \ltimes_{\alpha r} A/I$ is just $F(G, I)$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} G \ltimes_{\alpha r} A & \xrightarrow{\widetilde{\pi}} & G \ltimes_{\alpha r} A/I \\ {\scriptstyle R_\xi}\big\downarrow & & {\scriptstyle R_\xi}\big\downarrow \\ A & \xrightarrow{\pi} & A/I \end{array}$$

If $x \in G \ltimes_{\alpha r} A$ and $\widetilde{\pi}(x) = 0$ then we have $\pi(R_\xi(x)) = R_\xi(\widetilde{\pi}(x)) = 0$ for any $\xi \in L^2(G)$. Hence, we have $R_\xi(x) \in \ker \pi = I$ for each $\xi \in L^2(G)$ and so $x \in F(G, I)$. If $x \in F(G, I)$ then $R_\xi(\widetilde{\pi}(x)) = \pi(R_\xi(x)) = 0$ for each $\xi \in L^2(G)$ and so $\widetilde{\pi}(x) = 0$ by (3.2).

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