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A NOTE ON HOM(–,–) AS BCI–ALGEBRAS

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In 1966, K. Iséki [13] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Tiande and Changchang [16] discussed a new class of BCI-algebra, which is called a p-semisimple BCI-algebra. The class of p-semisimple BCI-algebras contains the class of associative BCI-algebras. Iséki and Thaheem [14] proved that if X is an associative BCI-algebra then $Hom(X)$, the set of all homomorphisms on X, is again an associative BCI-algebra. Aslam and Thaheem [1] proved that if X is a p-semisimple BCI-algebra then $Hom(X)$ is a p-semisimple BCI-algebra. Hoo and Murty [10] and Deeba and Goel [4] independently showed that $Hom(X)$ may not, in general, be a BCI-algebra for an arbitrarily BCI-algebra. In view of this result, we can also see that $Hom(X,Y)$, the set of all homomorphisms of a BCI-algebra X into an arbitrarily BCI-algebra Y, may not, in general, be a BCI-algebra. However, Deeba and Goel [4] proved that if X is a BCI-algebra and Y is a BCK-algebra, then $Hom(X,Y)$, the set of all homomorphisms from X to Y, is a BCK-algebra and hence a BCI-algebra. Liu [15] also showed the following:

PROPOSITION 1. *If X is a BCI-algebra and Y a p-semisimple BCI-algebra, then $Hom(X,Y)$ is a p-semisimple BCI-algebra.*

In this paper, we discuss the orthogonal subsets of BCI-algebras, and investigate their properties which are related to some ideals.

Recall that a BCI-algebra is an algebra $(X,*,0)$ of type (2, 0) satisfying the following conditions for all $x,y,z \in X$:

(1) $(x*y)*(x*z) \leq z*y$
(2) $x*(x*y) \leq y$
(3) $x \leq x$
(4) $x \leq y$ and $y \leq x$ imply $x=y$
(5) $x \leq 0$ implies $x=0$

where $x \leq y$ if and only if $x*y=0$.

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The following property holds in any BCI-algebra:

(6) $x * 0 = x$.

A BCI-algebra X is said to be associative [12] if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$. Let X_+ be the BCK-part of a BCI-algebra X, that is, X_+ is the set of all $x \in X$ such that $x \geq 0$. If $X_+ = \{0\}$, then X is called a p-semisimple BCI-algebra[16]. A non-empty subset I of a BCI-algebra X is called an ideal of X if (i) $0 \in I$, (ii) $y * x \in I$ and $x \in I$ imply that $y \in I$. A mapping $f : X \to Y$ between BCI-algebras X and Y is called a homomorphism if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$. Define the trivial homomorphism 0 as $0(x) = 0$ for all $x \in X$. Denote by $Hom(X, Y)$ the set of all homomorphisms of a BCI-algebra X into a BCI-algebra Y.

LEMMA 2. *([1], [2], [3], [5], [6], [11], [16]) Let X be a BCI-algebra. Then the following are equivalent:*

(7) *X is p-semisimple.*

(8) *$x * y = 0$ implies $x = y$.*

(9) *$x * a = x * b$ implies $a = b$.*

(10) *$a * x = b * x$ implies $a = b$.*

(11) *$a * (a * x) = x$.*

(12) *$0 * (0 * x) = x$.*

(13) *$0 * x = 0$ implies $x = 0$.*

(14) *$x * (0 * y) = y * (0 * x)$.*

(15) *$(x * y) * (w * z) = (x * w) * (y * z)$.*

Combining Proposition 1 and Lemma 2, we have:

PROPOSITION 3. *Let X, Y be BCI-algebras. If Y satisfies the one of (8) - (15), then $Hom(X, Y)$ is a p-semisimple BCI-algebra.*

In view of [16, Theorem 8 and Remark 2], we have the following:

PROPOSITION 4. *If X is a BCI-algebra and Y a p-semisimple BCI-algebra then $Hom(X, Y)$ is a quasi-commutative BCI-algebra of type (0, 1; 0, 0) and also of type (0, 2; 1, 0).*

We refer the reader to [7] for details on injective BCI-algebra.

PROPOSITION 5. *Let X and Y be BCI-algebras. If Y is injective then $Hom(X, Y)$ is a p-semisimple BCI-algebra.*

Proof. By [7], if an algebra is injective in the category of BCI-algebras then it is p-semisimple. It follows from Proposition 1 that $Hom(X,Y)$ is a p-semisimple BCI-algebra.

A non-empty subset I in a BCI-algebra X is a p-ideal of X [17] if
(16) $0 \in I$,
(17) $(x * z) * (y * z) \in I$ and $y \in I$ imply $x \in I$.

LEMMA 6. *([17]) A BCI-algebra X is p-semisimple if and only if every ideal of X is a p-ideal.*

LEMMA 7. *([17]) An ideal I of a BCI-algebra X is a p-ideal if and only if $(x * z) * (y * z) \in I$ implies $x * y \in I$, where $x, y, z \in X$.*

DEFINITION 8. Let X be a BCI-algebra and Y a p-semisimple BCI-algebra. Let M and Θ be subsets of X and $Hom(X,Y)$ respectively. We define orthogonal subsets $M^\perp$ and $\Theta^\perp$ of M and Θ respectively by

$$M^\perp = \{f \in Hom(X,Y) | f(x) = 0 \text{ for all } x \in M\}$$

and

$$\Theta^\perp = \{x \in X | f(x) = 0 \text{ for all } f \in \Theta\}.$$

PROPOSITION 9. *Let X and Y be BCI-algebras with $Y_+ = \{0\}$. Then we have the following:*
(18) $\{0\}^\perp = Hom(X,Y)$, *where 0 is the zero element of X.*
(19) $X^\perp = \{0\}$, *where 0 is the zero homomorphism.*
(20) *If $M_1 \subseteq M_2 \subseteq X$, then $M_2^\perp \subseteq M_1^\perp$.*
(21) $M \subseteq (M^\perp)^\perp$, *where $M \subseteq X$.*
(22) $M^\perp = ((M^\perp)^\perp)^\perp$, *where $M \subseteq X$.*
(23) $\{0\}^\perp = X$, *where 0 is the zero homomorphism.*
(24) $Hom(X,Y)^\perp = \{0\}$, *where 0 is the zero element of X.*
(25) *If $N_1 \subseteq N_2 \subseteq Hom(X,Y)$, then $N_2^\perp \subseteq N_1^\perp$.*
(26) $N \subseteq (N^\perp)^\perp$, *where $N \subseteq Hom(X,Y)$.*
(27) $N^\perp = ((N^\perp)^\perp)^\perp$, *where $N \subseteq Hom(X,Y)$.*

Proof. (18), (19), (23) and (24) follow easily from Definition 8.
(21) and (26) are easy.

(20) Assume that $M_1 \subseteq M_2 \subseteq X$. Let $f \in M_2^{\perp}$. Then $f(x) = 0$ for all $x \in M_2$. This implies $f(x) = 0$ for all $x \in M_1$, because $M_1 \subseteq M_2$. Hence $f \in M_1^{\perp}$ and $M_2^{\perp} \subseteq M_1^{\perp}$.

For (22) apply (26) to $M^{\perp}$ for $M^{\perp} \subseteq ((M^{\perp})^{\perp})^{\perp}$, and apply (20) to (21) for $((M^{\perp})^{\perp})^{\perp} \subseteq M^{\perp}$.

(25) and (27) are similar to that of (20) and (22) respectively.

THEOREM 10. *Let X be a BCI-algebra and Y a p-semisimple BCI-algebra. Let M and Θ be subsets of X and $Hom(X,Y)$ respectively. Then $M^{\perp}$ and $\Theta^{\perp}$ are ideals of $Hom(X,Y)$ and X respectively.*

Proof. Note that the zero homomorphism is contained in $M^{\perp}$. Let $f * g, g \in M^{\perp}$. Then for any $x \in M$, $0 = (f * g)(x) = f(x) * g(x) = f(x) * 0 = f(x)$. Thus $f \in M^{\perp}$, and so $M^{\perp}$ is an ideal of $Hom(X,Y)$. Next since $f(0) = 0$ for every $f \in \Theta$, we have $0 \in \Theta^{\perp}$. Assume that $y * x, x \in \Theta^{\perp}$. Then $0 = f(y * x) = f(y) * f(x) = f(y) * 0 = f(y)$ for every $f \in \Theta$. This implies that $y \in \Theta^{\perp}$, and that $\Theta^{\perp}$ is an ideal of X.

THEOREM 11. *$M^{\perp}$ and $\Theta^{\perp}$ are p-ideals of $Hom(X,Y)$ and X respectively.*

Proof. Since $M^{\perp}$ is an ideal, the fact that it is a p-ideal can be directly obtained from Proposition 1 and Lemma 6. But we prefer to give a direct proof. Note that $0 \in M^{\perp}$, where 0 is the zero homomorphism. Let $(f * h) * (g * h) \in M^{\perp}$ and $g \in M^{\perp}$. Then $0 = ((f * h) * (g * h))(x) = (f * h)(x) * (g * h)(x) = (f(x) * h(x)) * (g(x) * h(x))$ for any $x \in M$. Since Y is p-semisimple, it follows from (3), (6) and Lemma 2(15) that

$$\begin{aligned} 0 &= (f(x) * h(x)) * (g(x) * h(x)) \\ &= (f(x) * g(x)) * (h(x) * h(x)) \\ &= (f(x) * 0) * 0 \\ &= f(x) * 0 \\ &= f(x) \end{aligned}$$

for all $x \in M$. Thus $f \in M^{\perp}$ and $M^{\perp}$ is a p-ideal. Let us now prove that $\Theta^{\perp}$ is a p-ideal. By Lemma 7, it is enough to prove that $(x * z) * (y * z) \in \Theta^{\perp}$ implies $x * y \in \Theta^{\perp}$, where $x, y, z \in X$. Assume $(x * z) * (y * z) \in \Theta^{\perp}$

for every $x, y, z \in X$. Then by (3), (6) and (15), we have

$$\begin{aligned} 0 &= f((x*z)*(y*z)) \\ &= f(x*z)*f(y*z) \\ &= (f(x)*f(z))*(f(y)*f(z)) \\ &= (f(x)*f(y))*(f(z)*f(z)) \\ &= f(x)*f(y) \\ &= f(x*y) \end{aligned}$$

for any $f \in \Theta$. Thus $x*y \in \Theta^{\perp}$. This completes the proof.

The following corollary is obvious.

COROLLARY 12. *Let X and Y be BCI-algebras. If Y is injective, then $M^{\perp}$ and $\Theta^{\perp}$ are p-ideals of $Hom(X,Y)$ and X respectively.*

An ideal I of a BCI-algebra X is a closed ideal [9] if $0*x \in I$ whenever $x \in I$. It is said to be weakly implicative if whenever $(x*y)*z, y*z \in I$ then $(x*z)*z \in I$.

LEMMA 13. *([9]) If I is a closed ideal, then it is weakly implicative.*

THEOREM 14. *$M^{\perp}$ and $\Theta^{\perp}$ are closed ideals of $Hom(X,Y)$ and X respectively.*

Proof. We first show that $M^{\perp}$ is a closed ideal. It is enough to prove that $0*f \in M^{\perp}$ whenever $f \in M^{\perp}$. Let $f \in M^{\perp}$. Then for any $x \in M$, $(0*f)(x) = 0(x)*f(x) = 0*0 = 0$, which implies that $0*f \in M^{\perp}$. To prove that $\Theta^{\perp}$ is closed, it is sufficient to show that $0*x \in \Theta^{\perp}$ whenever $x \in \Theta^{\perp}$. Let $x \in \Theta^{\perp}$. Then $f(0*x) = f(0)*f(x) = 0*0 = 0$ for every $f \in \Theta$. Thus $0*x \in \Theta^{\perp}$. This completes the proof.

Combining Lemma 13 and Theorem 14, we have the following:

COROLLARY 15. *$M^{\perp}$ and $\Theta^{\perp}$ are weakly implicative ideals of $Hom(X, Y)$ and X respectively.*

The following corollary is obvious.

COROLLARY 16. *Let X and Y be BCI-algebras. If Y is injective, then $M^\perp$ and $\Theta^\perp$ are closed ideals and hence weakly implicative ideals of $Hom(X,Y)$ and X respectively.*

THEOREM 17. *Let X be a BCI-algebra, Y a p-semisimple BCI-algebra and $M \subseteq X$. Then $(M^\perp)^\perp$ is a p-ideal/a closed ideal of X containing M. Moreover, if M is a maximal p-ideal/a maximal closed ideal in X such that $M^\perp \neq \{0\}$, then $(M^\perp)^\perp = M$.*

Proof. By Theorems 11 and 14, $(M^\perp)^\perp$ is a p-ideal/a closed ideal of X, and by Proposition 9(21), $M \subseteq (M^\perp)^\perp$. The maximality of M implies that either $M = (M^\perp)^\perp$ or $(M^\perp)^\perp = X$. If $X = (M^\perp)^\perp$, then $f(x) = 0$ for every $x \in X$ and $f \in M^\perp$. Hence $f = 0$ for all $f \in M^\perp$. This gives $M^\perp = \{0\}$, a contradiction. Therefore $(M^\perp)^\perp = M$.

An ideal I of a BCI-algebra X is strongly implicative [9] if whenever $(x * y) * z \in I$ and $y * z \in I$, then $x \in I$.

THEOREM 18. *Let X be a BCI-algebra and Y an associative BCI-algebra. Then $M^\perp$ and $\Theta^\perp$ are strongly implicative ideals of $Hom(X,Y)$ and X respectively.*

Proof. We give the proof for $M^\perp$ and the proof for $\Theta^\perp$ will follow similarly. Let $(f * g) * h, g * h \in M^\perp$. Then $0 = ((f * g) * h)(x) = (f*g)(x)*h(x) = (f(x)*g(x))*h(x)$ and $0 = (g*h)(x) = g(x)*h(x)$ for any $x \in M$. Since Y is associative, it follows that $0 = (f(x)*g(x))*h(x) = f(x) * (g(x) * h(x)) = f(x) * 0 = f(x)$ for all $x \in M$ so that $f \in M^\perp$. This proves that $M^\perp$ is a strongly implicative ideal.

In view of [9, Proposition 1.1], we have the following corollary:

COROLLARY 19. *Let X be a BCI-algebra and Y an associative BCI-algebra. Then $X_+ \subset \Theta^\perp$.*

THEOREM 20. *Let X, Y and Z be BCI-algebras. If Z is p-semisimple, then to each homomorphism $f : X \to Y$ there corresponds a unique homomorphism $f^* : Hom(Y,Z) \to Hom(X,Z)$ that satisfies*

$$(*) \qquad f^*(g)(x) = (g \circ f)(x)$$

for all $x \in X$ and all $g \in Hom(Y,Z)$.

Proof. For each $g \in Hom(Y,Z)$ we can define a mapping $\mu : X \to Z$ by the relation $\mu(x) = g(f(x))$ for all $x \in X$. Since g and f are homomorphisms, therefore μ is a homomorphism and $\mu \in Hom(X,Z)$. Denote the function defined this way by $f^*(g) = \mu$. Thus $f^* : Hom(Y,Z) \to Hom(X,Z)$ is a mapping. To prove that f^* is a homomorphism, let $g, g' \in Hom(Y,Z)$. Then for any $x \in X$, $f^*(g*g')(x) = ((g*g')\circ f)(x) = (g*g')(f(x)) = g(f(x)) * g'(f(x)) = f^*(g)(x) * f^*(g')(x) = (f^*(g) * f^*(g'))(x)$. Since x is arbitrarily, it follows that $f^*(g*g') = f^*(g)*f^*(g')$ so that f^* is a homomorphism. The fact that (*) holds for all $x \in X$ obviously determines $f^*(g)$ uniquely. This completes the proof.

THEOREM 21. *Let X, Y and Z be BCI-algebras and let $f : X \to Y$ be a homomorphism. If Z is p-semisimple then $Ker(f^*) = Im(f)^{\perp}$ and $Ker(f) = Im(f^*)^{\perp}$.*

Proof. Let $\phi \in Ker(f^*)$. Then $f^*(\phi) = 0$ and hence $f^*(\phi)(x) = (\phi \circ f)(x) = 0$ for all $x \in X$. Thus $\phi \in Im(f)^{\perp}$ and $Ker(f^*) \subset Im(f)^{\perp}$. Similarly $Im(f)^{\perp} \subset Ker(f^*)$ and therefore $Ker(f^*) = Im(f)^{\perp}$. Next for any $\mu \in Im(f^*)$ we can find a homomorphism $g : Y \to Z$ such that $f^*(g) = \mu$. Then for any $x \in Ker(f)$, $\mu(x) = f^*(g)(x) = (g \circ f)(x) = g(f(x)) = g(0) = 0$, which implies that $x \in Im(f^*)^{\perp}$ and that $Ker(f) \subset Im(f^*)^{\perp}$. Conversely, let $x \in Im(f^*)^{\perp}$. Assume that $x \notin Ker(f)$, that is, $f(x) \neq 0$. Choose a homomorphism $g : Y \to Z$ with $g(f(x)) \neq 0$. If we say $f^*(g) = \mu$ for the g, then $\mu \in Im(f^*)$ and $\mu(x) = f^*(g)(x) = (g \circ f)(x) \neq 0$. This means that $x \notin Im(f^*)^{\perp}$ which is a contradiction. Thus $x \in Ker(f)$ and $Im(f^*)^{\perp} \subset Ker(f)$. This completes the proof.

The following corollary is obvious.

COROLLARY 22. *Let X, Y and Z be BCI-algebras. If Z is injective, then to each homomorphism $f : X \to Y$ there corresponds a unique homomorphism $f^* : Hom(Y,Z) \to Hom(X,Z)$ that satisfies $f^*(g)(x) = (g \circ f)(x)$ for all $x \in X$ and all $g \in Hom(Y,Z)$. Moreover, $Ker(f^*) = Im(f)^{\perp}$ and $Ker(f) = Im(f^*)^{\perp}$.*

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