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ON M_0–CONTINUOUS OPERATORS

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1. Intoduction

Throughout this paper X and Y are Banach spaces and μ is Lebesgue measure on $[0,1]$. $L^1(\mu)$ is a Banach space of all (classes of) Lebesgue integrable functions on $[0,1]$ with its usual norm. A bounded linear operator $T : L^1(\mu) \to X$ is (Bochner) representable if there is a bounded measurable function $g : [0,1] \to X$ such that $Tf = \int_{[0,1]} fg d\mu$ for all f in $L^1(\mu)$. A bounded linear operator $D : L^1(\mu) \to X$ is a Dunford-Pettis operator if D sends weakly compact sets into norm compact sets. A bound linear operator $T : L^1(\mu) \to X$ is nearly representable if $T \cdot D : L^1(\mu) \to X$ is Bochner representable for every Dunford-Pettis operator $D : L^1(\mu) \to L^1(\mu)$. For this oprator the followings are well-known facts [5].

Fact 1.1. Every representable operator $T : L^1(\mu) \to X$ is nearly representable. But the converse is not true.

Fact 1.2. Every nearly representable operator $T : L^1(\mu) \to X$ is Dunford-Pettis operator.

As a motive, the Volterra operator $V : L^1(\mu) \to C[0,1]$ defined by

$$Vf(t) = \int_{[0,t]} f d\mu, \quad a \le t \le 1$$

reveals many interesting properties. Bourgain showed that the Volterra oprator is nearly representable but not representable [1]. To show this he introduced the following M_0-norm.

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DEFINITION 1.3. The M_0-norm $|||\cdot|||_0$ on $L^1(\mu)$ is a real-valued function defined by

$$|||f|||_0 = \sup_I |\int_I f d\mu|, \quad f \in L^1(\mu)$$

where the supremum is taken over all subintervals I of $[0,1]$.

A sequence (f_n) in $L^1(\mu)$ is said to be M_0-convergent if it is convergent in this M_0-norm.

DEFINITION 1.4. A bounded linear operator $T : L^1(\mu) \to X$ is M_0-continuous if T is continuous for the M_0-norm on $L^1(\mu)$.

All the operators in this paper are assumed to be bounded and linear. The notations and symbols not appeared here can be seen in [2] and [3].

2. M_0-continuous Operators

It is obvious that every M_0-continuous operator is continuous. And it has intimate relations with nearly representable operators on $L^1(\mu)$. We can see the following fact in [4].

Fact 2.1. Every M_0-continuous operator $T : L^1(\mu) \to X$ is nearly representable.

Since every nearly representable operator implies Dunford-Pettis operator (Fact 1.2) the above fact tells us that every M_0-continuous operator is Dunford-Pettis operator. But on weakly compact subset of $L^1(\mu)$ these three operators are equivalent as we can see in the following proposition.

PROPOSITION 2.2. *Let K be a weakly compact subset of $L^1(\mu)$. If $T : L^1(\mu) \to X$ is a Dunford-Pettis operator, then $T_K : L^1(\mu) \to X$ defined by $T_K(f) = T(f\chi_K)$ is M_0-continuous.*

Proof. Let $(f_n\chi_K)$ be an M_0-convergent sequence in K. Since on weakly compact subset of $L^1(\mu)$ the M_0-norm agrees with the weak topology of $L^1(\mu)$, $(f_n\chi_K)$ is weakly convergent in $L^1(\mu)$. Since T is Dunford-Pettis operator $T(f_n\chi_K)$ converges in norm. Thus T is M_0-continuous.

The Example 10 of [4] shows that not every M_0-continuous operator on $L^1(\mu)$ is representable. Since all the weakly compact operators on $L^1(\mu)$ is representable, this means that not every M_0-continuous operator is weakly compact.

The proof of the following lemma can be seen in Theorem 9 of [4].

LEMMA 2.3. *If $T : L^1(\mu) \to X$ is M_0-continuous operator, then T can be factored through $C[0,1]$ with the Volterra operator.*

The next proposition plays an important role in investigating representability of bounded linear operators on $C[0,1]$ [3].

PROPOSITION 2.4. *Let $T : C[0,1] \to X$ be a bounded linear operator. Then there exists a weak* countably additive measure G defined on the Borel sets in $[0,1]$ with values in X^{**} such that*

(a) $G(\cdot)x^*$ is a regular countably additive Borel measure for each $x* \in X^*$,

(b) $x^*T(f) = \int_{[0,1]} f \, d(x^*G)$ for each $f \in C[0,1]$ and each $x^* \in X^*$ and

(c) $||T|| = ||G||([0,1])$.

It is well known that every weakly compact operator on $L^1(\mu)$ is representable but the converse is not true. But some representable operators are weakly compact. The following main theorem shows one criterion.

THEOREM 2.5. *If $T : L^1(\mu) \to X$ is M_0-continuous and representable, then T is weakly compact.*

Proof. Since T is M_0-continuous, by Lemma 2.3 T can be factored through $C[0,1]$ by Volterra operator $V : L^1(\mu) \to C[0,1]$ and bounded linear operator $L : C[0,1] \to X$ as $T = L \cdot V$. And the representability of T guarantees the existance of a $g \in L_\infty(\mu, X)$ such that

$$T(f) = \int fg \, d\mu = L \cdot V(f), \forall f \in L^1(\mu).$$

Let G be the representing measure of L, then by Proposition 2.4

$$\begin{aligned} x^* &\int_{[0,1]} fg \, d\mu \\ &= \int_{[0,1]} V(f) d(x^*G) \\ &= \int_{[0,1]} (\int_{[0,t]} f \, d\mu) d(x^*G)(t) \\ &= \int_{[0,1]} f(s) \int_{[s,1]} d(x^*G)(t) \, d\mu(s), \forall x^* \in X^*. \end{aligned}$$

Hence $\int_{[0,1]} fg \, d\mu = \int_{[0,1]} f(s) \int_{[s,1]} dG(t) d\mu(s)$ for every $f \in L^1(\mu)$. Thus

$$g(s) = \int_{[s,1]} dG(t) \ a.e.[\mu]$$

in X^{**}. Now let's show that $A = \{g(s) | s \in [0,1]\}$ is relatively weakly compact subset of X^{**}. Let $(g(s_n))$ be a sequence in A. Since (s_n) is a sequence in $[0,1]$, (s_n) has a convergent subsequence (s_{n_k}). Let (s_{n_k}) converges to $s \in [0,1]$. Since $x^*g(s_n) = \int_{[s_n,1]} d(x^*G)$ and x^*G is bounded regular from Proposition 2.4,

$$\chi_{[s_{n_k},1]} \to \chi_{[s,1]} \quad a.e.[x^*G].$$

So $(\int_{[s_{n_k},1]} dx^*G)$ converges to $\int_{[s,1]} dx^*G$. i.e., $x^*g(s_{n_k}) \to x^*g(s)$ for all $x^* \in X^*$. This implies $g(s_{n_k}) \to g(s)$ weak* in X^{**}. Hence $g(s_n) \to g(s)$ weakly in X. Thus g has essentially relatively weakly compact range. This implies the weakly compactness of $T : L^1(\mu) \to X$ [3].

REMARK 2.6. If we think of a quotient map $T : L^1(\mu) \to \ell_1$, then T is representable [3]. But since T is not weakly compact the Theorem 2.5 shows that T is not M_0-continuous. i.e., there exists a representable operator which is not M_0-continuous. Combined with this fact the Example 10 of [4] shows that there are no implications between M_0-continuous operators and representable operators on $L^1(\mu)$.

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