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JOINT DISTRIBUTION OF QUEUE LENGTH FOR TWO NODES QUEUEING NETWORK BY FUNCTIONAL EQUATIONS

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1. Introduction

In this paper we consider the two nodes queueing network model in which node 1 has general service time and node 2 has exponential service time (fig. 1). Customers arrive at the node 1 according to the Poisson process with rate λ. The node 1 has the service time distribution $G(\cdot)$ and node 2 has an exponential server with mean $\frac{1}{\mu}$. We assume that two nodes have waiting rooms of infinite capacity. Customers departing from each node may either leave the system or enter the another node according to Bernoulli schedule. Let $0 \leq p_i \leq 1, (i = 1,2)$ be the probability that the customer completing his service at node i enters the other node and let $q_i = 1 - p_i$, $i = 1,2$. When $p_1 = 1$ and $p_2 = 0$, our model becomes two nodes tandom queue denoted by $M/G/1 \rightarrow \cdot/M/1$ ([1]).

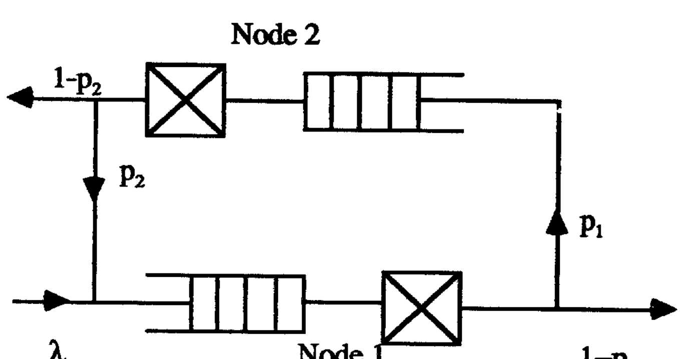

Fig. 1 Two nodes network

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When $0 < p_1 < 1$ and $p_2 = 1$, our model becomes an $M/G/1$ delayed feedback model. The $M/G/1$ delayed feedback model has been studied by many authors([4]). For the detailed list of related works in the queueing network refer the survey paper written by Disney and König [4]. Recently, Blanc et al. [1] obtained the closed form expression for the joint queue length distribution in the two node tandom queueing model with general service time at the first node and exponential distribution in the second node by solving the functional equation in two variables. The main purpose of this paper is to find the joint queue length distribution for our model. We show that the generating function $F(x, y)$ for the joint stationary queue length distribution in the model described above can be obtained by solving the functional equation of the following type

$$K(x,y)\Psi(x,y) = A(x,y)\Phi(x,y) + B(x,y)\Omega(y), \tag{1.1}$$

where Ψ, Φ and Ω are unknown and K, A and B are known functions. We solve the equation (1.1) and then give the expression for $F(x, y)$.

In section 2 we derive the equation (1.1) and study the properties of the equation $K(x, y) = 0$. In section 3, we solve the equation (1.1) in terms of $\Omega(y)$ and give the Fredholm integral equation of second kind for $\Omega(y)$. The generating function $F(x, y)$ is given in section 4.

2. The functional equation

Let $X_i(t)$ denote the number of customers present at node i ($i = 1, 2$) at time t, including the one being served, if any, and let $R(t)$ be the residual service time of the customer being served at node 1 at time t if $X_1(t) > 0$, otherwise $R(t) = 0$. Then the stochastic process $X = \{(X_1(t), X_2(t), R(t)),\ t \geq 0\}$ is a Markov process with state space $N \times N \times [0, +\infty)$, where N denotes the set of all nonnegative integers. We assume that the service time distribution $G(\cdot)$ at node 1 is not a lattice distribution and that $G(0+) = 0$. It is also assumed that the second order moment of the service times is finite. Let $\frac{1}{\nu}$ be the mean service time at node 1. Define for $t > 0$, $\tau > 0$, $i \geq 1$, $j \geq 0$,

$$p(t;i,j,\tau) = Pr(X_1(t) = i, X_2(t) = j, R(t) < \tau), \tag{2.1}$$

$$p(t;j) = Pr(X_1(t) = 0, X_2(t) = j), \tag{2.2}$$

$$q(t;i,j) = \lim_{\tau\to 0} \frac{\partial}{\partial\tau} p(t;i,j,\tau). \tag{2.3}$$

Through out this paper, we assume that the following conditions

$$(A1) \qquad \mu p_2 + \lambda < \nu,$$

$$(A2) \qquad \nu p_1 < \mu$$

hold. The conditions $(A1)$ and $(A2)$ guarantee the stability of node 1 and node 2, respectively. Hence under the conditions $(A1)$ and $(A2)$ the Markov process X possesses a unique stationary distribution. Let $p(i,j,\tau) = \lim_{t\to\infty} p(t;i,j,\tau)$ and $p(j) = \lim_{t\to\infty} p(t;j)$ and $q(i,j) = \lim_{t\to\infty} q(t;i,j)$. Considering the process transitions between t and $t+\Delta t$ and letting $\Delta t \to 0$ and then letting $t \to \infty$, we have the following set of differential equations for $i \geq 1$, $j \geq 0$, $\tau > 0$,

$$\begin{aligned}
-\frac{\partial}{\partial\tau}p(i,j,\tau) &= \lambda p(i-1,j,\tau)1_{i\geq 2} + \mu p_2 p(i-1,j+1,\tau)1_{i\geq 2} \\
&+ \lambda p(j)G(\tau)1_{i=1} + \mu p_2 p(j+1)G(\tau)1_{i=1} \\
&- (\lambda + \mu 1_{j\geq 1})p(i,j,\tau) - q(i,j) + \mu q_2 p(i,j+1,\tau) \\
&+ p_1 q(i+1,j-1)G(\tau)1_{j\geq 1} + q_1 q(i+1,j)G(\tau),
\end{aligned} \tag{2.4}$$

for $j \geq 0$

$$(\lambda + \mu 1_{j\geq 1})p(j) = \mu q_2 p(j+1) + p_1 q(1,j-1)1_{j\geq 1} + q_1 q(1,j), \tag{2.5}$$

where 1_A denotes the indicator function of the event A and $q_i = 1 - p_i$, $i = 1,2$. We introduce the following Laplace-Stieltjes transforms and generating functions;

$$\Xi(x,y,\sigma) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x^i y^j \int_0^{\infty} e^{-\sigma\tau} p(i+1,j,d\tau), \tag{2.6}$$

$$\Psi(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} q(i+1,j)x^i y^j, \tag{2.7}$$

$$\Omega(y) = \sum_{j=0}^{\infty} p(j)y^j, \tag{2.8}$$

$$F(x,y) = \lim_{t\to\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x^i y^j Pr(X_1(t) = i, X_2(t) = j), \tag{2.9}$$

for $|x| \le 1$, $|y| \le 1$ and $\mathrm{Re}(\sigma) \ge 0$. From (2.6)-(2.9), it is easily seen that the generating function $F(x, y)$ of the joint stationary queue length distribution satisfies the relation

$$F(x,y) = x\Xi(x,y,0) + \Omega(y), \tag{2.10}$$

for $|x| \le 1$, $|y| \le 1$. Multiplying equations (2.4) and (2.5) by $x^i y^j$and y^j and summing over i, j and j,respectively, we have the following relation for unknown functions $\Xi(x, y, \sigma)$, $\Psi(x, y)$ and $\Omega(y)$;

$$\begin{aligned} &x\Big(\lambda(1-x) + \mu(1 - \frac{p_2x+q_2}{y}) - \sigma\Big)\Xi(x,y,\sigma) \\ &= \mu(1 - \frac{p_2x+q_2}{y})\phi(x,\sigma) - (x - (p_1y+q_1)\beta(\sigma))\Psi(x,y) \\ &- \Big(\lambda(1-x) + \mu(1 - \frac{p_2x+q_2}{y})\Big)\beta(\sigma)\Omega(y) \end{aligned} \tag{2.11}$$

where $\phi(x, \sigma) = x\Xi(x, 0, \sigma) + \Omega(0)\beta(\sigma)$ and $\beta(\sigma) = \int_0^\infty e^{-\sigma t}dG(t)$. Letting $\sigma = \lambda(1-x) + \mu(1 - \frac{p_2x+q_2}{y})$, the right hand side of (2.11) must vanish and hence

$$K(x,y)\Psi(x,y) = A(x,y)\Phi(x,y) + B(x,y)\Omega(y), \tag{2.12}$$

for $|x| \le 1$, $|y| \le 1$, $\mathrm{Re}(\lambda(1-x) + \mu(1 - \frac{p_2x+q_2}{y})) \ge 0$ with

$$K(x,y) = x - (p_1y+q_1)\beta(\lambda(1-x) + \mu(1 - \frac{p_2x+q_2}{y})), \tag{2.13}$$

$$\Phi(x,y) = \phi(x, \lambda(1-x) + \mu(1 - \frac{p_2x+q_2}{y})), \tag{2.14}$$

$$A(x,y) = \mu(1 - \frac{p_2x+q_2}{y}), \tag{2.15}$$

$$\begin{aligned} B(x,y) = &-(\lambda(1-x) + \mu(1 - \frac{p_2x+q_2}{y})) \\ &\times \beta(\lambda(1-x) + \mu(1 - \frac{p_2x+q_2}{y})) \end{aligned} \tag{2.16}$$

The three unknown functions Ψ, Φ and Ω in (2.12) have the following properties;

- for every fixed $|y| \le 1$, $\Psi(x, y)$ is analytic for $|x| < 1$ and continuous for $|x| \le 1$, and similarly for x and y interchanged;
- for every fixed y with $|y| \ge 1$, $\Phi(x, y)$ is analytic in x for $|x| < 1$, continous for $|x| \le 1$;
- for every fixed x with $|x| \le 1$, $\Phi(x, y)$ is analytic in y for $|y| < 1$, continus for $|y| \ge 1$;
- $\Omega(y)$ is analytic for $|y| < 1$ and continuous for $|y| \le 1$.

The generating function $F(x, y)$ can be expressed in terms of $\Psi(x, y)$. Indeed, let $\sigma = 0$ in (2.11) and then for fixed $|x| \le 1$, choose y such that $\lambda(1-x) + \mu(1 - \frac{p_2x+q_2}{y}) = 0$ and $|y| \le 1$ i.e.

$$y = \theta(x) = \frac{(p_2x + q_2)\mu}{\lambda(1-x) + \mu}.$$

Then we have from (2.11) that

$$\phi(x, 0) = \theta(x)\frac{x - (p_1\theta(x) + q_1)}{\mu(\theta(x) - (p_2x + q_2))}\Psi(x, \theta(x)). \tag{2.17}$$

Letting $\sigma = 0$ in (2.11) and substuting (2.17) into (2.11), we have from the relation (2.10) that

(2.18)

$$F(x, y) = \frac{\mu(1 - \frac{p_2x+q_2}{y})\theta(x)\frac{x-(p_1\theta(x)+q_1)}{\mu(\theta(x)-(p_2x+q_2))}\Psi(x, \theta(x)) - (x - (p_1y + q_1))\Psi(x, y)}{\lambda(1-x) + \mu(1 - \frac{p_2x+q_2}{y})}.$$

Thus to find $F(x, y)$ it is enough to find the function $\Psi(x, y)$ for $|x| \le 1$ and $|y| \le 1$.

Now we consider the equation $K(x, y) = 0$.

LEMMA 2.1. *There exists only one root y_0 in $\{|y| > 1\}$ of the equation*

$$K(1, y) = 1 - (p_1y + q_1)\beta(\mu(1 - \frac{1}{y})) = 0. \tag{2.19}$$

For fixed y with $1 \le |y| \le y_0$, the equation $K(x,y) = 0$ has exactly one root $x = X(y)$ in $\{|x| \le 1\}$. All the roots have multiplicity one. Moreover, $|X(y)| = 1$ if and only if $y = 1$ or $y = y_0$, in this case $X(1) = X(y_0) = 1$.

Proof. The existence and uniqueness of y_0 in $\{|y| > 1\}$ are equivalent to the existence and uniqueness of the solution z_0 in $\{|z| < 1\}$ of the equation

$$z - (p_1 + q_1 z)\beta(\mu(1-z)) = 0.$$

Let $g(x) = (p_1 + q_1 x)\beta(\mu(1-x))$. Since g is convex on open interval (0,1) and $g(0) = p_1\beta(\mu)$ and $g'(1) = q_1 + \frac{\mu}{\nu} > 1$ by the condition ($A2$), $g(x) = x$ has a real solution z_0 in the interval (0,1) (see fig.2). Since for $|g(z)| \le g(|z|) < |z|$, for $1 > |z| > z_0$, by Rouche's theorem, z_0 is the unique solution of $g(z) = z$ in $\{|z| < 1\}$. Hence $y_0 = \frac{1}{z_0}(> 1)$ is the unique solution of (2.19) in $\{|y| > 1\}$.

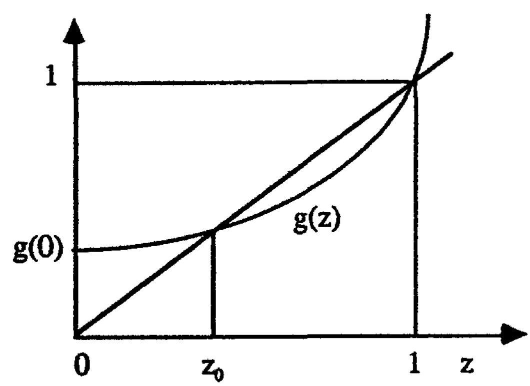

Fig. 2

Fix y with $|y| = 1$ and $y \ne 1$. Then, for $|x| = 1$, we have the following inequalities

$$\begin{aligned} &|(p_1 y + q_1)\beta(\lambda(1-x) + \mu(1 - \frac{p_2 x + q_2}{y}))| \\ <&|\beta(\lambda(1-x) + \mu(1 - \frac{p_2 x + q_2}{y}))| \\ <&1 \end{aligned}$$

By Rouche's theorem, for each y with $|y| = 1$, $y \neq 1$, the equation $K(x, y) = 0$ has exactly one solution in $\{|x| < 1\}$. From the relation $|\beta^{(1)}(s)| \leq \frac{|\beta(s)|}{\nu}$, we have the inequalities

$$
\begin{aligned}
|\frac{d}{dx}&\beta(\lambda(1-x) + \mu(1-(p_2x+q_2)))| \\
&= (\lambda + \mu p_2)|\beta^{(1)}(\lambda(1-x) + \mu(1-(p_2x+q_2)))| \\
&\leq (\lambda + \mu p_2)\frac{1}{\nu}|\beta(\lambda(1-x) + \mu(1-(p_2x+q_2)))| \\
&\leq \frac{\lambda + \mu p_2}{\nu} < 1
\end{aligned}
$$

for $|x| \leq 1$. Last inequality followed from the condition $A1$. Consequently, for all $|x| \leq 1$ with $x \neq 1$, we have

$$|1 - \beta(\lambda(1-x) + \mu(1-(p_2x+q_2)))| < |1-x|,$$

which therefore shows that $x = 1$ is the only solution of $K(x, 1) = 0$ in $\{|x| \leq 1\}$. Now we consider the case $1 < |y| \leq y_0$ and $y \neq y_0$. Note that for real y with $1 \leq y \leq y_0$, the inequality

$$(p_1 + \frac{q_1}{y})\beta(\mu(1 - \frac{1}{y})) \leq \frac{1}{y} \tag{2.20}$$

holds (see fig. 2), and the equality in the above inequality holds only the case $y = 1$ and $y = y_0$. For $|x| = 1$ and $1 < |y| \leq y_0$, $y \neq y_0$, we have

$$
\begin{aligned}
|(p_1 + &\frac{q_1}{y})\beta(\lambda(1-x) + \mu(1 - \frac{p_2x+q_2}{y}))| \\
&\leq (p_1 + \frac{q_1}{|y|})\beta(\mu(1 - \frac{1}{|y|})) \leq \frac{1}{|y|},
\end{aligned}
\tag{2.21}
$$

where the last inequality followed from (2.20). If y is not real, the first inequality in (2.21) is strict. The second inequality is strict except for $|y| = y_0$. Hence by Rouche's theorem, for $1 < |y| \leq y_0$ and $y \neq y_0$, the equation $K(x, y) = 0$ has exactly one solution in $\{|x| < 1\}$. By the same procedure $x = 1$ is only solution of $K(x, 1) = 0$, we can show that $x = 1$ is the only solution of $K(x, y_0) = 0$.

LEMMA 2.2. *$X(y)$ defined in lemma 2.1 is analytic in $\{1 \le |y| \le y_0,\ y \ne 1\}$ and continuous in $\{1 \le |y| \le y_0\}$.*

Proof. Fix y_1 with $1 \le |y_1| \le y_0$ and $y_1 \ne 1$, and define $x_1 = X(y_1)$. If $y_1 \ne 1$ and $y_1 \ne y_0$, then $|X(y_1)| < 1$. Thus it follows that there exist two real numbers $r_1 > 0$ and $r_2 > 0$ such that $Re(\lambda(1-x)+\mu(1-\frac{p_2x+q_2}{y})) > 0$ for $x \in \{|x - x_1| < r_1\}$ and $y \in \{|y - y_1| < r_2\}$. Now we consider the case $y_1 = y_0$. In this case take $r_2 = \frac{1}{2}(y_0 - 1) > 0$ and then take $r_1 = \frac{\mu r_2}{\lambda(y_0+1)+2\mu p_2} > 0$. Then $\mathrm{Re}(\lambda(1 - x) + \mu(1 - \frac{p_2x+q_2}{y})) > 0$ for $x \in \{|x - 1| < r_1\}$ and $\{|y - y_0| < r_2\}$. Hence $K(x,y)$ is analytic in $\{|x - x_1| < r_1\}$ for every fixed y with $|y - y_1| < r_2$ and analytic in $\{|y - y_1| < r_2\}$ for every fixed x with $|x - x_1| < r_1$. From the uniqueness of the solution x_1 and multiplicity one, we have

$$\frac{\partial}{\partial x}K(x,y)|_{(x_1,y_1)} \ne 0.$$

By the implicit function theorem for complex variables, $X(y)$ is analytic in $\{1 \le |y| \le y_0,\ y \ne 1\}$ and continuous in $\{1 \le |y| \le y_0\}$.

Let

$$C_0 = \{|y| = y_0\},\ C_0^+ = \{|y| < y_0\},\ C_0^- = \{|y| > y_0\}$$
$$L = \{X(y)|y \in C_0\}$$

and L^+(resp. L^-) denote the region on the left (resp. right) of the curve L, when moving on L in the counter clockwise. Because of the continuity of $X(y)$ on C_0 it is seen that L is a closed curve. By differentiating the relation $K(X(y),y) = 0$ with respect to y, we have

$$\begin{aligned}\frac{\partial}{\partial y}X(y) &= \frac{p_1\beta(\sigma(y)) + (\mu\frac{p_2X(y)+q_2}{y^2})(p_1y+q_1)\beta^{(1)}(\sigma(y))}{1+(\lambda+\mu p_2\frac{1}{y})(p_1y+q_1)\beta^{(1)}(\sigma(y))} \\ &= \frac{-\frac{\partial}{\partial y}K(x,y)|_{(X(y),y)}}{\frac{\partial}{\partial x}K(x,y)|_{(X(y),y)}},\end{aligned} \tag{2.22}$$

where $\sigma(y) = \lambda(1-X(y))+\mu(1-\frac{p_2X(y)+q_2}{y})$. Since $\frac{\partial}{\partial x}K(x,y)|_{(X(y),y)} \ne 0$, the denominator of $\frac{\partial}{\partial y}X(y)$ cannot vanish for $y \in C_0$, which shows that the curve L is everywhere differentiable. We easily see that $X^{(1)}(y) = 0$ if and only if $\frac{\partial}{\partial y}K(x,y)|_{(X(y),y)} = 0$, $y \in C_0$. We assume that the curve L is smooth and L^+ is simply connected domain, i.e.

$$(A3) \quad X^{(1)}(y) \neq 0, \text{ for } y \in C_0$$

$$(A4) \quad X(y_1) \neq X(y_2), \text{ for any } y_1,\ y_2 \in C_0 \text{ and } y_1 \neq y_2.$$

Following the procedure of Blanc et al. [1], we have the following propositions.

PROPOSITION 2.1. *For all $x \in \{|x| \leq 1\} \cap L^-$, the equation $K(x,y) = 0$ has exactly one root $y = Y(x)$ in $\{|y| > y_0\}$. Moreover, $Y(x)$ is analytic in $\{|x| < 1\} \cap L^-$ and continuous in $\{|x| \leq 1\} \cap L^-$ and can be analytically continued up to L and $Y(X(y)) = y$ for any $y \in C_0$.*

PROPOSITION 2.2. *For every $x \in L^+$, the equation $K(x,y) = 0$ has no roots in $\{|y| \geq y_0\}$.*

3. The Integral equation

Let us show that the functins $\Psi(x,y)$ and $\Omega(y)$ can be both analytically continued up to the contour C_0, for every fixed $|x| \leq 1$. Since $K(X(y),y) = 0$ for $|y| = 1$, we have from (2.12) that the following relation holds

$$\Omega(y) = -\frac{A(X(y),y)}{B(X(y),y)}\Phi(X(y),y), \quad |y| = 1. \tag{3.1}$$

We assert that the right hand side of (3.1) is analytic in $\{1 < |y| < y_0\}$ and continuous on $\{1 \leq |y| \leq y_0\}$. Indeed, $B(X(y),y) = 0$ implies that

$$\text{(i) } \lambda(1 - X(y)) + \mu(1 - \frac{p_2 X(y) + q_2}{y}) = 0, \text{ or}$$

$$\text{(ii) } \beta(\lambda(1 - X(y)) + \mu(1 - \frac{p_2 X(y) + q_2}{y})) = 0$$

If (i) holds, then $X(y) = \frac{(\lambda+\mu)y - \mu q_2}{\lambda y + \mu p_2}$ and hence

$$|X(y)| \geq \frac{(\lambda+\mu)|y| - \mu q_2}{\lambda |y| + \mu p_2} > 1,$$

for $1 < |y| \le y_0$. However since $|X(y)| \le 1$ for $1 < |y| \le y_0$, (i) cannot occur. If the case (ii) holds, it is easily seen from the definition of $X(y)$ that $X(y) = 0$. Hence $\Phi(X(y), y) = 0$.

Consequently, we deduce from the principle of analytic continuation that (3.1) gives the analytic continuaton of $\Omega(y)$ to $\{|y| \le y_0\}$. Note that

$$\Psi(x,y) = T_1(x,y)\Phi(x,y) + T_2(x,y)\Omega(y), \tag{3.2}$$

for $|x| \le 1$, $|y| \le 1$ and $\mathrm{Re}\,(\lambda(1-x)+\mu(1-\frac{p_2x+q_2}{y})) \ge 0$, where $T_1(x,y) = \frac{A(x,y)}{K(x,y)}$ and $T_2(x,y) = \frac{B(x,y)}{K(x,y)}$. Thus $\Psi(x,y)$ can be analytically continued to $\{|y| \le y_0\}$ for fixed x with $|x| \le 1$. Cauchy's integral formula for analytic function and the relation (3.2) yield the following relation; for $|x| \le 1$ with $x \notin L$ and $|y| < y_0$,

$$\Psi(x,y) = \frac{1}{2\pi i}\int_{C_0} \frac{T_1(x,t)}{t-y}\Phi(x,t)dt + \frac{1}{2\pi i}\int_{C_0} \frac{T_2(x,t)}{t-y}\Omega(t)dt. \tag{3.3}$$

Note that $K(x,y) \ne 0$ for $|x| \le 1$ with $x \notin L$ and $y \in C_0$, which ensures that both integrals in the right hand side of (3.3) are well-defined. By proposition 2.1 and 2.2, if $x \in \{|x| \le 1\} \cap L^-$, then $K(x,y)$ has exactly one zero $y = Y(x)$ in C_0^- and $K(x,y)$ has no zeros in $C_0^- \cup C_0$ if $x \in L^+$. This entails that the function $t \mapsto T_1(x,t)\Phi(x,t)$ is analytic in C_0^- and continuous in $C_0^- \cup C_0$ for any $x \in L^+$, and that it has exactly one pole at $y = Y(x)$ in C_0^- for any $x \in \{|x| \le 1\} \cap L^-$. Thus for $x \in L^+$ and $y \in C_0^+$, we have that

$$\frac{1}{2\pi i}\int_{C_0} \frac{T_1(x,t)}{t-y}\Phi(x,t)dt = 0 \tag{3.4}$$

and for $x \in \{|x| \le 1\} \cap L^-$,

$$\frac{1}{2\pi i}\int_{C_0} \frac{T_1(x,t)}{t-y}\Phi(x,t)dt = \frac{\Gamma(x)}{Y(x)-y},$$

where $\Gamma(x)$ is the residue of the function $t \mapsto T_1(x,t)\Phi(x,t)$ at $t = Y(x)$, i.e.

(3.5)

$$\begin{aligned}\Gamma(x) &= \lim_{t\to Y(x)} \frac{t-Y(x)}{K(x,t)} A(x,t)\Phi(x,t)\\ &= \frac{A(x,Y(x))\Phi(x,Y(x))}{\frac{\partial}{\partial t}K(x,t)|_{t=Y(x)}}\\ &= -\frac{\mu(1-\frac{p_2x+q_2}{Y(x)})\Phi(x,Y(x))}{p_1\beta(\sigma(Y(x))+(p_1Y(x)+q_1)\frac{\mu}{Y(x)^2}(p_2x+q_2)\beta^{(1)}(\sigma(Y(x)))}.\end{aligned}$$

LEMMA 3.1. *The function $\Gamma(x)$ is continuous in $\{|x| \le 1\} \cap (L^- \cup L)$.*

Proof. Recall that $Y(x)$ is a continuous and non-vanishing function for $x \in \{|x| \le 1\} \cap (L^- \cup L)$. Since, for each $x \in \{|x| \le 1\} \cap L^-$, the solution of $K(x,y) = 0$ is simple, we have $\frac{\partial}{\partial y}K(x,y)|_{y=Y(x)} \ne 0$. Noting the following equivalences

$$(A3) \Leftrightarrow \frac{\partial}{\partial y}K(x,y)|_{(X(y),y)} \ne 0,\ y \in C_0 \Leftrightarrow \frac{\partial}{\partial y}K(x,y)|_{(x,Y(x))} \ne 0,\ x \in L,$$

the lemma is proved.

From (3.5) and (3.1), we have

$$\begin{aligned}\Gamma(X(y)) &= -\Omega(y)\frac{-B(X(y),y)}{\frac{\partial}{\partial y}K(x,y)|_{(X(y),y)}}\\ &= -Q(y)\Omega(y),\ y \in C_0,\end{aligned} \tag{3.6}$$

where

$$Q(y) = \frac{\sigma(y)\beta(\sigma(y))}{p_1\beta(\sigma(y)) + (p_1y+q_1)\frac{\mu}{y^2}(p_2X(y)+q_2)\beta^{(1)}(\sigma(y))}.$$

Let

$$\Pi(x,y) = \frac{1}{2\pi i}\int_{C_0} \frac{T_2(x,t)}{t-y}\Omega(t)dt.$$

Then (3.3) becomes
(3.7)

$$\Psi(x,y)=\begin{cases}\Pi(x,y) & \text{for } x\in L^+,\ |y|<y_0\\ \Pi(x,y)+\frac{\Gamma(x)}{Y(x)-y} & \text{for } x\in\{|x|\le 1\}\cap L^-,\ |y|<y_0.\end{cases}$$

Let $Z(x)=p_2x+q_2$, $|x|\le 1$. Then clearly $|Z(x)|\le 1$ for $|x|\le 1$. We have from (2.15) that $A(x,Z(x))=0$. From (2.12) we have

$$\Psi(x,Z(x))=T_2(x,Z(x))\Omega(Z(x)). \tag{3.8}$$

From (3.7) we have for $x\in\{|x|\le 1\}\cap L^-$,

$$\Psi(x,Z(x))=\Pi(x,Z(x))+\frac{\Gamma(x)}{Y(x)-Z(x)}. \tag{3.9}$$

We have from (3.8), (3.9) and (3.7) that for $x\in\{|x|\le 1\}\cap L^-$,

$$\begin{aligned}\Gamma(x)&=(Z(x)-Y(x))\{\Pi(x,Z(x))-T_2(x,Z(x))\Omega(Z(x))\}\\ &=\frac{Z(x)-Y(x)}{2\pi i}\int_{C_0}\frac{T_2(x,t)-T_2(x,Z(x))}{t-Z(x)}\Omega(t)dt.\end{aligned} \tag{3.10}$$

Note that $T_2(x,Z(x))=\frac{B(x,Z(x))}{K(x,Z(x))}$ is analytic in $\{|x|<1\}$ and continuous on $\{|x|\le 1\}$. Indeed, let $h(x)=(p_1p_2x+p_1q_2+q_1)\beta(\lambda(1-x))$. Then $K(x,Z(x))=x-h(x)$. Note that $h(x)$ is convex on the interval (0,1) and $h(0)=(p_1q_2+q_1)\beta(\lambda)>0$ and $h'(1)=\frac{1}{\nu}(\lambda+p_2\nu p_1)<1$ by (A1) and (A2). Thus we have $|h(x)|\le h(|x|)<|x|$ for $|x|<1$, and hence by Rouche's theorem the equation $K(x,Z(x))=0$ has no solution in $\{|x|<1\}$. Since $|h(x)|<1$ for $|x|=1$, $x\ne 1$, the unique zero of $K(x,Z(x))$ is $x=1$. However $B(x,Z(x))$ has also zero at $x=1$. Thus $T_2(x,t)-T_2(x,Z(x))$ has a pole at $x=X(t)$, $t\in C_0$. Then the integral (3.10) is singular integral when $x\in L$. The following lemma helps us to remove this singularity.

LEMMA 3.2. *For any $z\in C_0-\{y_0\}$, there exists a neighborhood V_z of the point z such that*
1)$X(y)$ is analytic in V_z
2)$X(V_z\cap C_0^-)\subset\{|x|\le 1\}\cap L^-$

3)$X(V_z \cap C_0^+) \subset L^+$
4)$V_z \subset \{|y| > 1\}$.

Proof. See Blanc et al. [1].

Let

$$V = \cup_{z \in C_0 - \{y_0\}} V_z$$

and define for $y \in V$, $t \in C_0$,

$$H(y,t) = \frac{T_2(X(y),t) - T_2(X(y), Z(X(y)))}{t - Z(X(y))} - \frac{\Lambda(y)}{t-y}, \tag{3.11}$$

where $\Lambda(y)$ is the residue of the function

$$t \mapsto \frac{T_2(X(y),t) - T_2(X(y), Z(X(y)))}{t - Z(X(y))}$$

at $t = y$. In fact, for any $y \in V$,

$$\begin{aligned}\Lambda(y) &= \lim_{t \to y} \frac{(t-y)}{K(X(y),t)} \frac{B(X(y),t)}{t - X(X(y))} \\ &= \frac{1}{y - Z(X(y))} \frac{B(X(y),y)}{\frac{\partial}{\partial t} K(X(y),t)|_{t=y}} \\ &= \frac{Q(y)}{y - Z(X(y))}.\end{aligned}$$

LEMMA 3.3. *The function* $H(y,t)$ *posseses the following properties:*
1) for fixed $y \in V$, *the mapping* $t \mapsto H(y,t)$ *is continuous on the circle* C_0
2) for fixed $t \in C_0$, *the mapping* $y \mapsto H(y,t)$ *is continuous in* V.

Proof. (1). If $y \in V - C_0$, then the continuity of $t \mapsto H(y,t)$ on C_0 readily follows from definition (3.11). Similary if $y \in C_0$, then $H(y,t)$ is continuous on $C_0 - \{y\}$. It remains to prove that $H(y,t)$ has a finite limit whenever $t \to y$ if $y \in C_0$. The existence of this limit follows from the fact $\Lambda(y)$ is the residue of the function $t \mapsto \frac{T_2(X(y),t)-T_2(X(y),Z(X(y)))}{t-Z(X(y))}$ at the point $t = y \in C_0$. Tedious calculation yields the limit

$$H_1(y) = \lim_{\substack{t\to y \\ t\in C_0-\{y\}}} H(y,t)$$

(3.12)
$$= -\frac{T_2(X(y),Z(X(y)))}{y-Z(X(y))} + II,$$

where

$$II = -\frac{\Lambda(y)}{y-Z(X(y))} + \frac{III}{y-Z(X(y))},$$

$$III = \Big(p_1\beta(\sigma(y)) + (p_1y+q_1)(\frac{\mu}{y^2}(p_2X(y)+q_2))\beta^{(1)}(\sigma(y))\Big)^{-1}$$

$$\times \mu\frac{p_2X(y)+q_2}{y^2}\Big(\beta(\sigma(y)) + \sigma(y)\beta^{(1)}(\sigma(y))\Big) + \sigma(y)\beta(\sigma(y))\cdot(IV),$$

$$IV = \frac{q_1\mu}{y}(p_2X(y)+q_2)\beta^{(1)}(\sigma(y)) + \frac{1}{2}(p_1y+q_1)\Big(\frac{\mu}{y^2}(p_2X(y)+q_2)\Big)^2\beta^{(2)}(\sigma(y)).$$

(2). Fix now $t \in C_0$. Then, clearly $y \mapsto H(y,t)$ is continuous in $V - \{t\}$ from definition (3.11). It remains to prove that $H(y,t)$ has a finite limit whenever $y \to t$. After tedious calculation we have

$$H_2(t) = \lim_{\substack{y\to t \\ y\in V-\{t\}}} H(y,t)$$

(3.13)
$$= -\frac{T_2(X(t),Z(X(t)))}{t-Z(X(t))} + I,$$

where

$$I = \Lambda^{(1)}(t) + \frac{p_2X^{(1)}(t)\Lambda(t)}{p_2X(t)-t+q_2} + \frac{II}{p_2X(t)-t+q_2},$$

$$II = -\Big(X^{(1)}(t) + (p_1t+q_1)(\lambda+\frac{\mu p_2}{t})X^{(1)}(t)\beta^{(1)}(\sigma(t))\Big)^{-1}$$

$$\times (\lambda+\frac{\mu p_2}{t})X^{(1)}(t)\Big(\beta(\sigma(t)) + \sigma(t)\beta^{(1)}(\sigma(t))\Big)$$

$$- B(X(t),t)\cdot III,$$

$$III = -\frac{1}{2}\Big(X^{(2)}(t) + (p_1t+q_1)(\lambda+\frac{\mu p_2}{t})X^{(2)}(t)\beta^{(1)}(\sigma(t))$$

$$(p_1t+q_1)\Big((\lambda+\frac{\mu p_2}{t})X^{(1)}(t)\Big)^2\beta^{(2)}(\sigma(t))\Big).$$

From (3.10) and (3.11) we have for $z \in V \cap C_0^-$,

(3.14)

$$\begin{aligned}\Gamma(X(z)) =& \frac{Z(X(z))-z}{2\pi i}\int_{C_0}\Omega(t)H(z,t)dt + \frac{Z(X(z))-z}{2\pi i}\int_{C_0}\frac{\Lambda(z)}{t-z}\Omega(t)dt\\ =& \frac{Z(X(z))-z}{2\pi i}\int_{C_0}\Omega(t)H(z,t)dt.\end{aligned}$$

Letting $z \to y \in C_0$ with $z \in V \cap C_0^-$, we have from (3.14) that for $y \in C_0$

$$\Gamma(X(y)) = \frac{Z(X(y))-y}{2\pi i}\int_{C_0}\Omega(t)H^*(y,t)dt, \tag{3.15}$$

where

$$H^*(y,t) = \begin{cases} H(y,t) & \text{if } t \neq y \\ H_2(y) & \text{if } t = y. \end{cases}$$

By combining (3.6) and (3.15), we derive the following integral equation:

$$\Omega(y) = \int_{C_0} N(y,t)\Omega(t)dt, \; y \in C_0 \tag{3.16}$$

where

$$N(y,t) = -\frac{(Z(X(y))-y)H^*(y,t)}{2\pi i Q(y)}. \tag{3.17}$$

REMARK. For every $t \in C_0$, the function $y \mapsto N(y,t)$ is continuous on C_0. Similarily the function $t \mapsto N(y,t)$ is continuous on $C_0 - \{y\}$ and

$$\lim_{t\to y} N(y,t) = -\frac{Z(X(y))-y}{2\pi i Q(y)}H_1(y),$$

which is finite. Hence we have

$$\int_{C_0}\int_{C_0}|N(y,t)|^2 dydt < \infty. \tag{3.18}$$

Thus (3.15) defines a homogeneous Fredholm integral equation of the second kind on the circle C_0.

4. The generating function

PROPOSITION 4.1. *The real part of $\Omega(y)$ on the circle C_0 is given as the unique up to additive constant nonzero and continuous solution of the following homogeneous Fredholm integral equation of the second kind*

$$\Omega_R(y) = \int_0^{2\pi} \Omega_R(t) N_R(y,t)\, dt, \qquad t = y_0 e^{i\varphi}, \qquad y \in C_0, \tag{4.1}$$

where

$$\Omega_R(y) = Re(\Omega(y)), \tag{4.2}$$

$$N_R(y,t) = 2Re(itN(y,t)). \tag{4.3}$$

Proof. The proof of proposition 4.1 is essentially the same as the proof of proposition 5.1 of [1].

Let us now show that the knowledge of $\Omega_R(y)$ on C_0 is actually sufficient for determine the generating function $F(x,y)$. If $\Omega_1(t)$ and $\Omega_2(t)$ be two solutions of (4.1), then $\Omega_1(t) = \Omega_2(t) + C$, for all $t \in C_0$, where C is a constant.

Let $\Omega_0(t)$ be a solution of (4.1) with $\Omega_0(y_0) = 0$. Then the required solution is

$$\Omega_R(t) = \Omega_0(t) + C, \tag{4.4}$$

where $C = \Omega_R(y_0)$. The constant C is determined later. For $|w| = 1$ and $t = y_0 w$, we clearly have

$$\begin{aligned}\Omega(t) &= 2Re(\Omega(t)) - \bar{\Omega}(t)\\ &= 2Re(\Omega(t)) - \Omega(\bar{t})\\ &= 2\Omega_R(t) - \Omega(\frac{y_0}{w}),\end{aligned}$$

since the coefficients of $\Omega(y)$ are all real numbers. Thus the function $\Pi(x,y)$ defined in section 3 can be rewritten as follows; for $|x| \le 1$ with

$x \notin L$ and $|y| < y_0$,

$$
\begin{aligned}
\Pi(x,y) =& \frac{1}{\pi i}\int_{C_0} \frac{T_2(x,t)}{t-y}\Omega_R(t)dt - \frac{1}{2\pi i}\int_{C_0} \frac{T_2(x,t)}{t-y}\Omega(\bar{t})dt \\
=& \frac{1}{\pi i}\int_{C_0} \frac{T_2(x,t)}{t-y}\Omega_R(t)dt,
\end{aligned} \tag{4.5}
$$

from the Cauchy's theorem, since the function

$$
w \mapsto \frac{T_2(x,t)}{t-y}\Omega(\frac{y_0}{w}),\ t = y_0 w
$$

is analytic for $\{|w| > 1\}$ and continuous for $\{|w| \geq 1\}$ for fixed x, y with $|x| \leq 1$, $x \notin L$ and $|y| < y_0$. Consequently,

(4.6)

$$
\begin{aligned}
\Psi(x,y) &= \begin{cases} \frac{1}{\pi i}\int_{C_0} \frac{T_2(x,t)}{t-y}\Omega_R(t)dt, & \text{for } x \in L^+ \\ \frac{1}{\pi i}\int_{C_0} \frac{T_2(x,t)}{t-y}\Omega_R(t)dt & \\ -\frac{Z(x)-Y(x)}{y-Y(x)}\frac{1}{\pi i}\int_{C_0} \frac{T_2(x,t)-T_2(x,Z(x))}{t-Z(x)}\Omega_R(t)dt, & \text{for } x \in \{|x| \leq 1\} \cap L^- \end{cases} \\
&= \Psi_0(x,y) + C\Psi_1(x,y), \text{ for } |y| < y_0,
\end{aligned}
$$

where

(4.7)

$$
\Psi_0(x,y) = \begin{cases} \frac{1}{\pi i}\int_{C_0} \frac{T_2(x,t)}{t-y}\Omega_0(t)dt, & \text{for } x \in L^+ \\ \frac{1}{\pi i}\int_{C_0} \frac{T_2(x,t)}{t-y}\Omega_0(t)dt & \\ -\frac{Z(x)-Y(x)}{y-Y(x)}\frac{1}{\pi i}\int_{C_0} \frac{T_2(x,t)-T_2(x,Z(x))}{t-Z(x)}\Omega_0(t)dt, & \text{for } x \in \{|x| \leq 1\} \cap L^- \end{cases}
$$

and

(4.8)

$$
\Psi_1(x,y) = \begin{cases} \frac{1}{\pi i}\int_{C_0} \frac{T_2(x,t)}{t-y}dt, & \text{for } x \in L^+ \\ \frac{1}{\pi i}\int_{C_0} \frac{T_2(x,t)}{t-y}dt & \\ -\frac{Z(x)-Y(x)}{y-Y(x)}\frac{1}{\pi i}\int_{C_0} \frac{T_2(x,t)-T_2(x,Z(x))}{t-Z(x)}dt, & \text{for } x \in \{|x| \leq\} \cap L^- \end{cases}
$$

Now we determine the constant C using the fact $F(1,1)=1$. By substituting (4.6) into (2.18), we have

$$
\begin{aligned}
F(x,y) = &\Big(\lambda(1-x)+\mu(1-\frac{p_2x+q_2}{y})\Big)^{-1}\\
&\times\Big(\mu(1-\frac{p_2x+q_2}{y})\theta(x)\frac{x-(p_1\theta(x)+q_1)}{\mu(\theta(x)-(p_2x+q_2))}\\
&(\Psi_0(x,\theta(x))+C\Psi_1(x,\theta(x)))\\
&-(x-(p_1y+q_1))(\Psi_0(x,y)+C\Psi_1(x,y))\Big).
\end{aligned}
\tag{4.9}
$$

Letting $x\to 1$ in (4.9), it is easily obtained that

$$
\begin{aligned}
F(1,y) =&\frac{1}{\lambda}(1-p_1p_2-p_1\frac{\lambda}{\mu})(\Psi_0(1,1)+C\Psi_1(1,1))\\
&-\frac{y(1-p_1y-q_1)}{\mu(y-1)}(\Psi_0(1,y)+C\Psi_1(1,y)).
\end{aligned}
\tag{4.10}
$$

Letting $y\to 1$ in (4.10), we have

$$
1=\Big(\frac{1}{\mu}(1-p_1p_2-p_1\frac{\lambda}{\mu})+\frac{p_1}{\mu}\Big)(\Psi_0(1,1)+C\Psi_1(1,1))
$$

and hence

$$
C=\frac{1-\frac{1}{\lambda}(1-p_1p_2)\Psi_0(1,1)}{\frac{1}{\lambda}(1-p_1p_2)\Psi_1(1,1)}.
\tag{4.11}
$$

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