Comm. Korean Math. Soc. 8 (1993), No. 1, pp. 67–76

NUMERICAL SOLUTIONS OF THE PLATEAU PROBLEM BY FINITE DIFFERENCE METHODS

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1. Introduction

The Plateau problem is one of the most interesting mathematical problems. The problem is to construct a surface S of least area. Let $u(x, y)$ represent the height of S in a simply connected region D. Then the surface area of S is equal to

$$J(u) = \iint_D (1 + u_x^2 + u_y^2)^{\frac{1}{2}}\, dx\, dy, \tag{1.1}$$

and the Plateau problem is to find a function $u(x, y)$ which minimizes the functional $J(u)$ with satisfying the boundary condition

$$u(x, y) = f(x, y), \qquad (x, y) \in \partial D, \tag{1.2}$$

where ∂D is the boundary of the domain D.

Since the Euler–Lagrange equation of (1.1) which minimizes $J(u)$ is

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0, \tag{1.3}$$

the Plateau problem is equivalent to solve the partial differential equation (1.3) with boundary condition (1.2). This implies that the solution $u(x, y)$ of (1.1) must be a single valued function with continuous second partial derivatives.

The existence and uniqueness of the problem (1.1)–(1.2) has been studied by Stepleman [8] and numerical solutions of the Plateau problem have been studied by Greenspan [6] and Concus [3] by using finite

Received April 20, 1992. Revised August 27, 1992.
This research is partially supported by KOSEF under Grant 891–0106–009–1.

difference methods. But their methods can be applied only to the problems with single valued boundary conditions. In 1974, Hinata, Shimasaki and Kiyono [7] used a finite element method to solve the Plateau problem with multiple valued problems. They converted the multiple valued problems to free boundary value problems to overcome the restriction on boundary conditions. Tschuchiya [9] has also studied the Plateau problem in parametric form and used finite element methods to obtain numerical solutions.

In this paper, a computational method to solve the Plateau problem with multiple valued boundary conditions by finite difference methods will be presented. In the following section, the multiple valued boundary conditions will be changed into single valued free boundary conditions along the idea of [7]. In section 3, to use finite difference methods in a non-rectangular domain we imply a transformation to obtain a rectangular domain. And we discretize the transformed minimizing functional. Finally, some numerical experiments are given. In [9], they used the largest eigenvalue of Hessian matirx of (1.1) in SOR, but it may cause the difficulty to find the largest eigenvalue if the size of Hessian matrix is large. We used the maximum value of the second derivatives in place of the largest eigenvalue of SOR which is used in [9], so we don't have to calculate the largest eigenvalue of the Hessian matrix.

2. Free Boundary Value Problem

We consider the Plateau problem of finding a twice continuously differentiable function $u(x,y)$ which minimizes the surface area functional $J(u)$ with boundary conditions

$$\begin{aligned}
&\frac{\partial y}{\partial x} = 0, \qquad x = 0, \quad 0 \le u \le L, \\
&x = r\sin\theta, \qquad y = 0, \quad 0 \le u \le \frac{L}{2}, \\
&x^2 + (y + r\cos\theta)^2 = r^2, \qquad u = 0, \quad 0 \le x, \ 0 \le y, \\
&\frac{\partial x}{\partial u} = 0, \qquad u = 0, \quad 0 \le y \le H,
\end{aligned} \tag{2.1}$$

where r and L are the given positive real numbers and H is an unknown distance from the u-axis to the free boundary at $u = \frac{L}{2}$.

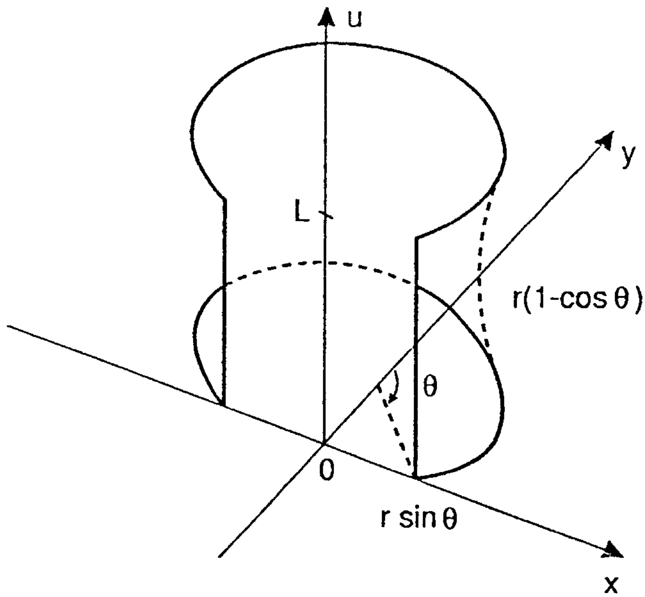

Figure 2.1

Since the problem (1.1) and (2.1) has multiple valed boundary conditions, it cannot be solved numerically by the methods introduced by Greenspan [6] and Concus [3]. But it can be changed into a free boundary value problem with single valued boundary conditions. This free boundary value problem is obtained by rotating axes. By the symmetry with respect to $u = \frac{L}{2}$, the problem becomes to find a function $x(y,u)$ which minimizes the functional on the rotated domain $\mathcal{D}$

$$J(x) = \iint_{\mathcal{D}} (1 + x_y^2 + x_u^2)^{\frac{1}{2}}\, dydu, \tag{2.2}$$

with boundary conditions

$$\begin{aligned} & x = r\sin\theta, \qquad y = 0, \quad 0 \le u \le \frac{L}{2}, \\ & x^2 + (y + r\cos\theta)^2 = r^2, \qquad u = 0, \quad 0 \le x, \quad 0 \le y, \\ & \frac{\partial x}{\partial u} = 0, \qquad u = \frac{L}{2}, \quad 0 \le y \le H, \\ & \frac{\partial y}{\partial x} = 0, \qquad x = 0, \quad 0 \le u \le \frac{L}{2}. \end{aligned} \tag{2.3}$$

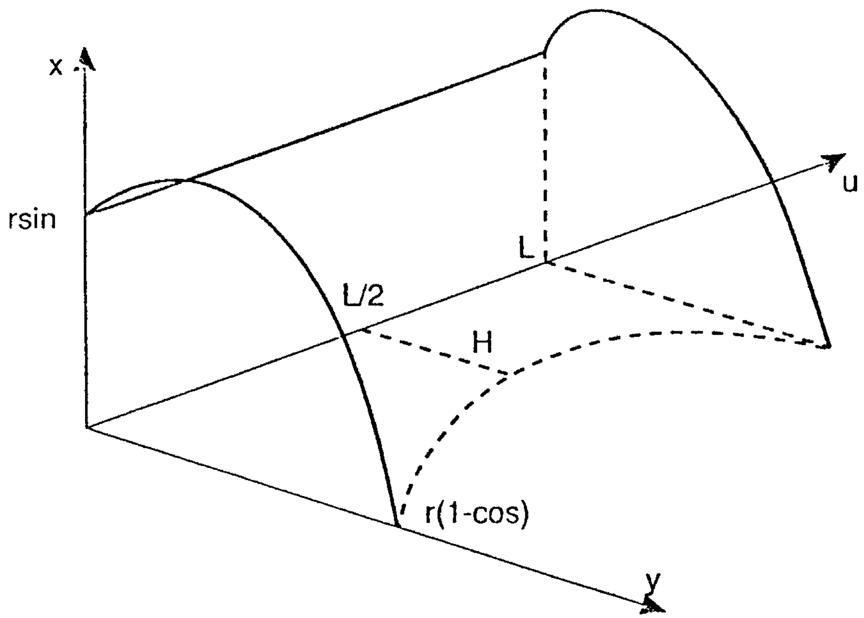

Figure 2.2

3. A Discrete Area Functional

Let M and N be positive integers and $\Delta y = \frac{r(1-\cos\theta)}{M}, \Delta u = \frac{L}{2N}$. Let y_{Mj} be the largest boundary value on the free boundary in y component at $u_j = j\Delta u$. And define the interior points y_{ij} of the rotated domain as

$$y_{ij} = \frac{y_{i0}y_{Mj}}{y_{M0}}, \quad i = 0,1,\ldots,M, \quad j = 0,1,\ldots,N.$$

Since the domain is not rectangular, it is not easy to apply finite difference methods for approximations of (1.1) and (2.1). But by the transformation T defined by

$$T: \begin{cases} \xi(y,u) &= \frac{y\, y_{M0} \Delta u}{(u-u_j)(y_{Mj+1}-y_{Mj})+\Delta u\, y_{Mj}}, \quad j\Delta u \le u < (j+1)\Delta u, \\ \eta(y,u) &= u, \qquad 0 \le u \le \frac{L}{2}, \end{cases}$$

then the transformed domain $T(D)$ becomes a rectangular one.

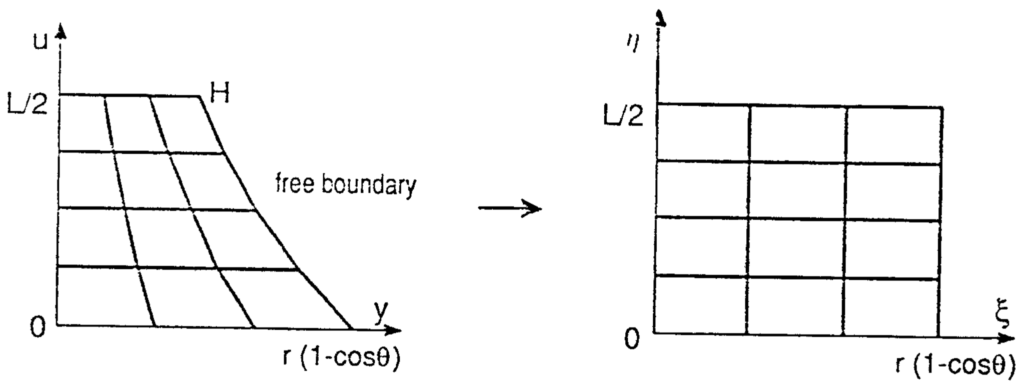

Figure 3.1

Using the forward difference schemes for x_ξ and x_η, we obtain a discrete area functional

(3.1)

$$J_\Delta(X) = \sum_{i=1}^{M}\sum_{j=1}^{N}\{y_{Mj}^2 + y_{M0}^2[\frac{x_{i+1j} - x_{ij}}{\Delta\xi}]^2$$

$$+ [\frac{x_{i+1j} - x_{ij}}{\Delta\xi}\frac{-\xi_i(y_{Mj+1} - y_{Mj})}{\Delta\eta y_{Mj}} + y_{Mj}\frac{x_{ij+1} - x_{ij}}{\Delta\eta}]^2\}^{\frac{1}{2}}\frac{1}{y_{M0}}\Delta\xi\Delta\eta.$$

Thus we obtain the following reduced problem:

Find a stationary point of functional

$$F(X) = J_\Delta(X) : R^n \to R, \tag{3.2}$$

where $n = M(N+1), X = (x_1, x_2, \ldots, x_{MN}, x_{MN+1}, \ldots, x_n)$ and $x_{MN+1}, \ldots, x_n$

are the values of the unknown boundary.

In order to find stationary points of $F(X)$ numerically, we may use the nonlinear successive overrelaxation(NSOR) method

$$x_i^{k+1} = x_i^k - \omega \frac{F_i(X_i^k)}{F_{ii}(X_i^k)}, \tag{3.3}$$

where $X_i^k = (x_1^{k+1}, \ldots, x_{i-1}^{k+1}, x_i^k, \ldots, x_n^k)$, $F_i(X) = \frac{\partial F(X)}{\partial x_i}$, $F_{ii}(X) = \frac{\partial^2 F(X)}{\partial x_i^2}$, and ω is a relaxation parameter. However, if the denominator $F_{ii}(X_i^k)$ is small, ω has to be chosen sufficiently small. We therefore introduce a modified NSOR method. Let

$$\bar{x}_i^{k+1} = x_i^k - \alpha_{ik} \frac{F_i(X_i^k)}{m_{ik}}, \tag{3.4}$$

$$x_i^{k+1} = x_i^k + \omega_{ik}(\bar{x}_i^{k+1} - x_i^k), \tag{3.5}$$

where $\alpha_{ik} < 2$ and $m_{ik} = \max_{X \in I_{ik}} |F_{ii}(X)|, I_{ik} = \{X \in Dom(F) : F(X) \leq F(X_i^k)\}$. Then from the relations (3.4) and (3.5), we obtain

$$x_i^{k+1} = x_i^k - \alpha_{ik}\omega_{ik} \frac{F_i(X_i^k)}{m_{ik}}, \quad i = 1, 2, \ldots, n. \tag{3.6}$$

Tschuchiya [9] used similar methods to get numerical solutions of the Plateau problem by finite element methods. He used the largest eigenvalue of the Hessian matrix instead of m_{ik}.

Before we discuss the convergence of the sequence $\{X^k\}$ generated by (3.6), note that the functional $F(X)$ is twice continuously differentible strictly convex and bounded below. And assume that the domain of F is convex and the set $S_0 = \{X \in Dom(F) : F(X) \leq F(X^0)\}$ is nonempty and compact for some $X^0 \in Dom(F)$.

THEOREM 1. *Assume that there exists a constant $\beta > 0$ such that $m_{ik} \geq \beta$ for every $X \in S_0$. Then the sequence generated by (3.6)* *converges to* $X^* \in S_0$ *and* X^* *is a stationary point of* $F(X)$ *if we choose* ω_{ik} *so that*

$$\alpha_{ik}\omega_{ik} \leq \frac{2}{1 + \frac{\delta_{ik}|F_i(X_i^k)|}{m_{ik}^2}}$$

for some positive real number δ_{ik}.

Proof. It follows from the Taylor expansion of $F(X)$ that
(3.7)

$$\begin{aligned} F(X_i^{k+1}) &= F(X_i^k) + F_i(X_i^k)(X_i^{k+1} - X_i^k) + \frac{1}{2}F_{ii}(\Xi)(X_i^{k+1} - X_i^k)^2 \\ &= F(X_i^k) - \alpha_{ik}\omega_{ik}\frac{F_i(X_i^k)^2}{m_{ik}} + \frac{1}{2}\alpha_{ik}^2\omega_{ik}^2 F_{ii}(\Xi)\frac{F_i(X_i^k)^2}{m_{ik}^2}, \end{aligned}$$

where $\Xi = (\xi_1, \xi_2, \ldots, \xi_n)$ is a vector between X_i^k and X_i^{k+1} and $\Xi = X_i^k$ execpt ξ_i. Since $F(X)$ is twice continuously differentiable on a compact set S_0, we may choose a positive constant δ_{ik} such that

$$|F_{ii}(\Xi) - F_{ii}(X_i^k)| \leq \delta_{ik}\frac{|F_i(X_i^k)|}{m_{ik}}, \quad \text{whenever} \quad |\Xi - X_i^k| \leq 2\frac{|F_i(X_i^k)|}{m_{ik}}.$$

Since $|\Xi - X_i^k| \leq 2\frac{|F_i(X_i^k)|}{m_{ik}}$ from (3.4) and (3.5), it follows from (3.6) and hypothesis for $\delta_{ik}\omega_{ik}$ that

$$\begin{aligned} F(X_i^{k+1}) - F(X_i^k) &\leq -\alpha_{ik}\omega_{ik}\frac{F_i(X_i^k)^2}{m_{ik}^2}\{m_{ik} - \frac{\alpha_{ik}\omega_{ik}}{2}F_{ii}(X_i^k)\} \\ &\quad + \frac{1}{2}\alpha_{ik}^2\omega_{ik}^2\frac{F_i(X_i^k)^2}{m_{ik}^2}\{F_{ii}(\Xi) - F_{ii}(X_i^k)\} \\ &\leq -\alpha_{ik}\omega_{ik}\frac{F_i(X_i^k)^2}{m_{ik}^2}m_{ik}[1 - \frac{\alpha_{ik}\omega_{ik}}{2}(1 + \frac{\delta_{ik}|F_i(X_i^k)|}{m_{ik}^2})] \\ &\leq 0. \end{aligned}$$

Thus, from the process of obtaining $\{X_i^k\}_k$, the sequence $\{F(X_i^k)\}_k$ is decreasing for $i = 1, \ldots, n$. Since $F(X)$ is bounded below, $\{F(X_i^k)\}$ converges.

It follows from (3.7) that

$$
\begin{aligned}
F(X_i^k) - F(X_i^{k+1}) &= \frac{m_{ik}}{\alpha_{ik}\omega_{ik}}(X_i^{k+1} - X_i^k)^2 - \frac{1}{2}F_{ii}(\Xi)(X_i^{k+1} - X_i^k)^2 \\
&\geq (X_i^{k+1} - X_i^k)^2 m_{ik}\left(\frac{1}{\alpha_{ik}\omega_{ik}} - \frac{1}{2}\right) \\
&\geq 0.
\end{aligned}
$$

Hence the sequence $\{X^k\}$ converges to X^* in S_0 by the convergence of $\{F(X_i^k)\}$, and X^* is clearly a stationary point of $F(X)$.

We now give some numerical results for the problem (1.1) and (2.1) with $r = 1.0, \theta = \frac{5}{6}\pi, M = 15$, and $\Delta\eta = 0.05$. For the maximum values m_{ik}, in (3.6), we simply used $\max\{|F_{ii}(X_i^k)|, |F_{ii}(\bar{X}_i^{k+1})|\}$. The iterations were terminated when $|x_i^k - x_i^{k+1}| \leq 10^{-5}$ as in [7]. The following table 1 and 2 show the numerical results of the problem obtained by using algorithm (3.6) with initial values of $\alpha = 0.1$ and 0.2, respectively. The computation was carried out by using IBM 3090 at Seoul National University.

L/2	no. of iterations	H	1/4 of area
0.4	1589	1.80112	1.02277
0.5	2453	1.75213	1.26133
0.6	11278	1.59628	1.48180
0.7	*		

* (diverges with $\alpha = 0.1$)

Table 1

L/2	no. of iterations	H	1/4 of area
0.4	945	1.79072	1.02280
0.5	4267	1.71008	1.26044
0.6	*		

* (diverges with $\alpha = 0.2$)

Table 2

The table 3 and 4 show the values of maximum relaxation factor $\alpha_k \omega_k$ for the calculation of x–values and the free boundary values. We used 0.1 and 0.2 for the initial values of α, respectively.

L/2	x-values	boundary values
0.4	0.11358	0.19886
0.5	0.11599	0.13950
0.6	0.11184	0.13865
0.7	*	*

* (diverges with $\alpha = 0.1$)

Table 3

L/2	x-values	boundary values
0.4	0.22241	0.04205
0.5	0.22603	0.10782
0.6	*	*

* (diverges with $\alpha = 0.2$)

Table 4

REMARKS. The method (3.3) with fixed $\omega = 0.1$ diverges when $L/2 = 0.6, M = 15$, and $\Delta\eta = 0.05$ as in the table 1. NSOR method with variable relaxation parameter has been studied in [1]–[2].

ACKNOWLEDGEMENT. The author would like to thank the refree for valuable criticisms.

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