Comm. Korean Math. Soc. 8 (1993), No. 1, pp. 55–66

ON KOORNWINDER'S GENERALIZED JACOBI POLYNOMIALS

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1. Introduction

The problem of classifying all differential equations of the form

$$\sum_{i=0}^{2r}\sum_{j=0}^{i}\ell_{ij}x^j y^{(i)}(x)=\lambda_n y(x) \tag{1.1}$$

having an orthogonal polynomial sequence(OPS) as solutions has attracted much interest over the last fifty years, since it provides new examples of self-adjoint differential operators illustrating the theory of singular boundary value problems of higher order differential equations initiated by Weyl and Titchmarsh ([3,8,11]).

In 1929, Bochner([1]) solved the problem for $r=1$ and found that there are only 4 distinct OPS's of Jacobi, Laguerre, Hermitte and Bessel up to a linear change of the variable. In 1938, H. L. Krall([9]) classified all fourth order equations and discovered three new nonclassical OPS's, which are later named as the Legendre type, Laguerre type and Jacobi type polynomials by A. M. Krall([7]).

In 1984, Koornwinder([6]) found the polynomial sequences $\{P_n^{\alpha,\beta,M,N}(x)\}_0^\infty$ which are orthogonal with respect to the weight function $(1-x)^\alpha(1+x)^\beta+M\delta(x+1)+N\delta(x-1)$, $\alpha,\beta>-1$ and $M,N\geq 0$. As a limiting case, he found the generalized Laguerre polynomials $\{L_n^{\alpha,N}(x)\}_0^\infty$ which are orthogonal with respect to the weight function $\frac{1}{\Gamma(\alpha+1)}x^\alpha e^{-x}+N\delta(x),\alpha>-1$ and $N\geq 0$. Recently, J. Koekoek and R. Koekoek([5]) showed that $\{L_n^{\alpha,N}(x)\}_0^\infty$ satisfy a differential equation of infinite order of the form

$$N\sum_{0}^{\infty}a_i(x)y^{(i)}(x)+xy''+(\alpha+1-x)y'+ny=0, \tag{1.2}$$

Received April 6, 1992.

where all the $a_i(x)$'s are polynomials of degree less than or equal to i for each i and all except $a_0(x)$ are independent of n. In particular, they showed that if α is a nonnegative integer, the order of the differential equation (1.2) is $2\alpha+4$.

However it is not known in general whether the generalized Jacobi polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_0^\infty$ satisfy a differential equation of the form (1.1) except the following three special cases.
When $M=N=0$, $\{P_n^{\alpha,\alpha,0,0}(x)\}_0^\infty$ is just the Jacobi polynomials satisfying a second order differential equation of the form (1.1).
When $\alpha=\beta=0$ and $M=N$, $\{P_n^{0,0,M,M}(x)\}_0^\infty$ is the Legendre type polynomials(found by H. L. Krall([10])) satisfying a fourth order differential equation of the form (1.1).
When $\alpha=\beta=0$ and $M\neq N$, $\{P_n^{0,0,M,N}(x)\}_0^\infty$ is the Krall polynomials (found by L. L. Littlejohn[12]) satisfying a sixth order differential equation of the form (1.1).

In this work we construct differential equations satisfied by Koornwinder's generalized Jacobi polynomials $\{P_n^{\alpha,\alpha,M,M}(x)\}_0^\infty$ for $\alpha=\beta=1,2,3$ and $M=N$ and give a conjecture for a differential equation satisfied by $\{P_n^{\alpha,\alpha,M,M}(x)\}_0^\infty$ and its order when M is positive. Also we give a generating function and a Rodrigues' type formula for $\{P_n^{\alpha,\alpha,M,M}(x)\}_0^\infty$.
This work is partially supported by KOSEF(Grant No. 90-08-00-02) and GARC. We are grateful to the refree's kind revising the original manuscript.

2. Differential Equations for $\alpha=\beta=1,2,3$ and $\mathbf{M}=\mathbf{N}$

In this section we give the explicit expressions of differential equations satisfied by $\{P_n^{\alpha,\alpha,M,M}(x)\}_0^\infty$ for $\alpha=1,2,3$.

DEFINITION 2.1. Let I be an open interval on the real line and $\{a_i(x)\}_0^N$ real valued functions in $C^i(I)$ for each $i=0,1,\cdots,N$. Then the differential operator $L=\sum_0^N a_i(x)(\frac{d}{dx})^i$ is called symmetric if $L=L^*$ where L^* is the formal adjoint of L given by

$$L^*=\sum_0^N(-1)^i(a_iy)^{(i)}.$$

A function $s(x)(\not\equiv 0)$ in $C^N(I)$ is called a symmeric factor of L if sL becomes symmetric.

Now a symmetric factor of a differential operator is characterized in the next lemma(see [13] for the proof).

LEMMA 2.2. *Let $L = \sum_0^{2r} a_i(x)(\frac{d}{dx})^i$ be a differential operator as in Definition 2.1. Then for a function $s(x) \not\equiv 0$ in $C^{2r}(I)$ the following statements are equivalent.*

(a) *$s(x)$ is a symmetric factor of L, that is, $sL = (sL)^*$.*
(b) *$s(x)$ simultaneously satisfies the following system of r homogeneous equations, called the symmetric equations:*

(2.1)
$$\sum_{i=k}^{r} \binom{2i}{2k-1} \frac{2^{2i-2k+2}-1}{i-k+1} B_{2i-2k+2}(a_{2i}(x)s(x))^{(2i-2k+1)} - a_{2k-1}s = 0$$

for $k = 1, 2, \cdots, r$ where B_{2i} are Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!} x^{2i}.$$

Now let $\{P_n(x)\}_0^\infty$ be an OPS satsfying

$$L_{2r}(P_n)(x) = \sum_0^{2r} \sum_{j=0}^{i} \ell_{ij} x^j P_n^{(i)}(x) = \lambda_n P_n(x) \tag{2.2}$$

where $\lambda_n = \ell_{00} + n\ell_{11} + n(n-1)\ell_{22} + \cdots + n(n-1)\cdots(n-2r+1)\ell_{2r,2r}$. Then the symmetric equations for L_{2r} yield an orthogonalizing weight for $\{P_n(x)\}_0^\infty$ when they are solved not only classically but also in some generalized function spaces of distributions ([8]) or hyperfunctions([4]). More precisely we have:

LEMMA 2.3([8]). *Let $\{P_n(x)\}_0^\infty$ be an OPS satisfying (2.2). If $\{P_n(x)\}_0^\infty$ is orthogonal relative to a distribution Λ acting on polynomials, then Λ must satisfy*

$$\langle \Lambda, \sum_{i=k}^{r} \binom{2i}{2k-1} \frac{2^{2i-2k+2}-1}{i-k+1} B_{2i-2k+2} a_{2i}(x)\phi(x)^{(2i-2k+1)} - a_{2k-1}(x)\phi(x)\rangle = 0 \tag{2.3}$$

for any polynomial $\phi(x)$ and $k = 1, 2, \cdots, r$.

Now consider the orthogonalizing weight $\Lambda = (1-x^2)^\alpha + M\delta(x-1) + M\delta(x+1)$ for $\{P_n^{\alpha,\alpha,M,M}(x)\}_0^\infty$ and its absolutely continuous part $s(x) = (1-x^2)^\alpha$. We will derive differential equations of the form (2.2) satisfied by $\{P_n^{\alpha,\alpha,M,M}(x)\}_0^\infty$ for $\alpha = 0, 1, 2, 3$. The construction of differential equations depends on the following two ideas:

(i) The differential operator of the form (2.2) having an OPS as solutions can be determined if the orthogonalizing weight is known and the corrresponding moment sequence satisfies a certain set of recurrence relations([9]).

(ii) For all known differential equations of the form (2.2) having an orthogonal polynomial solutions, the absolutely continuous part of orthogonalizing weight serves as the symmetric factor of the differetial equation([8]).

If a symmetric factor $s(x)$ is sufficiently smooth, then we have from (2.1)

$$a_{2k-1}(x) = \frac{1}{s(x)} \sum_{i=k}^{r} \binom{2i}{2k-1} \frac{2^{2i-2k+2}-1}{i-k+1} B_{2i-2k+2} (a_{2i}(x)s(x))^{(2i-2k+1)} \tag{2.4}$$

for $k = 1, 2, \cdots, r$.

We illustrate our method by exemplifying the case $\alpha = 0$ only. Suppose that $\{P_n^{0,0,M,M}(x)\}_0^\infty$ satisfy a fourth order differential equation of the form (2.2).

From (2.4), we have

$$a_3(x) = 2a_4'(x), \tag{2.5}$$

$$a_1(x) = -a_4^{(3)}(x) + a_2'(x). \tag{2.6}$$

From (2.3), we have

$$
\begin{aligned}
0 &= \langle \Lambda, 2a_4(x)\phi'(x) + a_3(x)\phi(x) \rangle \\
&= 2Ma_4(-1)\phi'(-1) + 2Ma_4(1)\phi'(1) \\
&\quad + [-2a_4(-1) + Ma_3(-1)]\,\phi(-1) + [-2a_4(1) + Ma_3(1)]\,\phi(1),
\end{aligned} \tag{2.7}
$$

(2.8)

$$
\begin{aligned}
0 &= \langle \Lambda, -a_4(x)\phi^{(3)}(x) + a_2(x)\phi'(x) + a_1(x)\phi(x) \rangle \\
&= -Ma_4(-1)\phi^{(3)}(-1) - Ma_4(1)\phi^{(3)}(1) + a_4(-1)\phi''(-1) - a_4(1)\phi''(1) \\
&\quad + [-a_4'(-1) + Ma_2(-1)]\,\phi'(-1) + [-a_4'(1) + Ma_2(1)]\,\phi'(1) \\
&\quad + [a_4''(-1) - a_2(-1) + Ma_1(-1)]\,\phi(-1) \\
&\quad + [a_4''(1) - a_2(1) + Ma_1(1)]\,\phi(1).
\end{aligned}
$$

Since (2.7) and (2.8) are satisfied for all polynomials $\phi(x)$, we can obtain sufficiently many equations to determine the coefficients of $a_2(x)$ and $a_4(x)$. They are

$$
\begin{aligned}
&a_4(-1) = a_4(1) = 0; \quad a_3(-1) = 2a_4'(-1) = 0; \quad a_3(1) = 2a_4'(1) = 0; \\
&a_2(-1) = a_2(1) = 0; \quad a_4''(-1) + Ma_1(-1) = a_4''(1) + Ma_1(1) = 0; \\
&a_4''(-1) + M(-a_4^{(3)}(-1) + a_2'(-1)) = 0; \\
&a_4''(1) + M(-a_4^{(3)}(1) + a_2'(1)) = 0.
\end{aligned}
$$

Thus we have $a_4(x) = \ell_{44}(x^2-1)^2$ and $a_2(x) = \ell_{22}(x^2-1)$. If we set $\ell_{44} = M$, we find that $\ell_{22} = 4+12M$ and the desired differential equation is given by

$$
\begin{aligned}
&M\left[(x^2-1)^2 y^{(4)} + 8x(x^2-1)y^{(3)} + 12(x^2-1)y''\right] \\
&\qquad + 4(x^2-1)y'' + 8xy' = \lambda_n y
\end{aligned} \tag{2.9}
$$

where $\lambda_n = M\,[n(n-1)(n-2)(n-3) + 8n(n-1)(n-2) + 12n(n-1)]$ $+4n(n-1) + 8n$.

Note that this agrees with H. L. Krall's result([10]) and further it can be written in the form

$$
\begin{aligned}
&M\left[\frac{1}{4}(x^2-1)^2y^{(4)}+2x(x^2-1)y^{(3)}+3(x^2-1)y''\right.\\
&\left.\quad+\frac{1}{4}(n-1)n(n+1)(n+2)y\right]\\
&\quad+(x^2-1)y''+2xy'+n(n+1)y=0
\end{aligned}
\tag{2.10}
$$

where the last three terms is the differential equation satisfied by Jacobi polynomials $\{P_n^{(0,0)}(x)\}_0^\infty$.

Suppose that $\{P_n^{\alpha,\alpha,M,M}(x)\}_0^\infty$ satisfy a $(2\alpha+4)$th order differential equation of the form (2.2). By the same method as above we can also find the corresponding differential equations of the form (2.2) for the case $\alpha=1,2,3$. We list them below.

When $\alpha=1$, $P_n^{1,1,M,M}(x)$ satisfies

$$
\begin{aligned}
&M\left[(x^2-1)^3y^{(6)}+24x(x^2-1)^2y^{(5)}+60(x^2-1)(3x^2-1)y^{(4)}\right.\\
&\left.\quad+480x(x^2-1)y^{(3)}+360(x^2-1)y^{(2)}\right]+48\left[(x^2-1)y''+6xy'\right]=\lambda_n y.
\end{aligned}
$$

When $\alpha=2$, $P_n^{2,2,M,M}(x)$ satisfies

$$
\begin{aligned}
&M\left[(x^2-1)^4y^{(8)}+48x(x^2-1)^3y^{(7)}+168(x^2-1)^2(5x^2-1)y^{(6)}\right.\\
&\quad+1344x(x^2-1)(5x^2-3)y^{(5)}+5040(x^2-1)(5x^2-1)y^{(4)}\\
&\left.\quad+40320x(x^2-1)y^{(3)}+20160(x^2-1)y''\right]\\
&\quad+1536[(x^2-1)y''+6xy']=\lambda_n y.
\end{aligned}
$$

When $\alpha=3$, $P_n^{3,3,M,M}(x)$ satisfies

$$
\begin{aligned}
&M\left[(x^2-1)^5y^{(10)}+80x(x^2-1)^4y^{(9)}+360(x^2-1)^3(7x^2-1)y^{(8)}\right.\\
&\quad+5760x(x^2-1)^2(7x^2-3)y^{(7)}+10080(x^2-1)(35x^4+30x^2-3)y^{(6)}\\
&\quad+241920x(x^2-1)(7x^2-3)y^{(5)}+604800(x^2-1)(7x^2-1)y^{(4)}\\
&\left.\quad+4838400x(x^2-1)y^{(3)}+1814400(x^2-1)y''\right]\\
&\quad+100800[(x^2-1)y''+8xy']=\lambda_n y.
\end{aligned}
$$

Note that all these differential equations can also be written in the form (3.1) (cf. section 3).

3. Differential equations for general case

From the result in section 2, we guess that $\{P_n^{\alpha,\alpha,M,M}(x)\}_0^\infty$ will satisfy a differential equation of the form

(3.1)
$$M\sum_0^\infty a_i(x)y^{(i)}(x)+(x^2-1)y''(x)+(2\alpha+2)xy'(x)-n(n+2\alpha+1)y(x)=0.$$

The Koornwinder's generalized Jacobi polynomials $\{P_n^{\alpha,\alpha,M,M}(x)\}_0^\infty$ are given by (see[6])

(3.2)
$$P_n^{\alpha,\alpha,M,M}(x)=\left[1+\frac{Mn(2\alpha+2)_n}{(\alpha+1)n!}\right]\cdot \left[1+\frac{M(2\alpha+1)_n}{(2\alpha+1)n!}\{-2x\frac{d}{dx}+\frac{n(M+2\alpha+1)}{(2\alpha+1)}\}\right]P_n^{(\alpha,\alpha)}(x),$$

where $P_n^{(\alpha,\alpha)}(x)$ are the Jacobi polynomials satisfying

$$(x^2-1)y''+(2\alpha+2)xy'-n(n+2\alpha+1)y=0 \tag{3.3}$$

and $(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)$.

We want to determine $\{a_i(x)\}_0^\infty$ such that $\{P_n^{\alpha,\alpha,M,M}(x)\}_0^\infty$ satisfy a differential equation (3.1). To do this, we set $y(x)=P_n^{(\alpha,\alpha)}(x)$ in (3.2) and use the definition and the Jacobi equation (3.3) to find that

$$\begin{aligned}
&M\left[\sum_0^\infty a_i(x)D^iP_n^{(\alpha,\alpha)}(x)+\frac{4(2\alpha+1)_n}{(2\alpha+1)n!}D^2P_n^{(\alpha,\alpha)}(x)\right]\\
&\quad+\frac{M^2(2\alpha+1)_n}{(2\alpha+1)n!}\left[\frac{n(n+2\alpha+1)}{\alpha+1}\sum_0^\infty a_i(x)D^iP_n^{(\alpha,\alpha)}(x)\right.\\
&\quad\left.-2\sum_0^\infty a_i(x)\{xD^{i+1}+iD^i\}\,P_n^{(\alpha,\alpha)}(x)\right]=0
\end{aligned} \tag{3.4}$$

for $M \geq 0$ and for all $x \in [-1,1]$.
Since the expressions in the square brackets are independent of M, this implies that

$$\sum_0^\infty a_i(x)D^iP_n^{(\alpha,\alpha)}(x) + \frac{4(2\alpha+2)_{n-1}}{n!}D^2P_n^{(\alpha,\alpha)}(x) = 0 \tag{3.5}$$

and

(3.6)
$$\frac{n(n+2\alpha+1)}{\alpha+1}\sum_0^\infty a_i(x)D^iP_n^{(\alpha,\alpha)}(x) - 2\sum_0^\infty a_i(x)(xD^{i+1}+iD^i)P_n^{(\alpha,\alpha)}(x) = 0.$$

Using (3.5), we can reduce (3.6) to

$$\sum_0^\infty a_i(x)(xD^{i+1}+iD^i)P_n^{(\alpha,\alpha)}(x) + \frac{4(2\alpha+3)_{n-1}}{(n-1)!}D^2P_n^{(\alpha,\alpha)}(x) = 0. \tag{3.7}$$

Then we have a system of equations (3.5) and (3.7) for $\{a_i(x)\}_0^\infty$.

Multiplying (3.5) by n and substracting (3.7) we obtain

(3.8)
$$a_0(n,\alpha)\left[nP_n^{(\alpha,\alpha)}(x) - xDP_n^{(\alpha,\alpha)}(x)\right]$$
$$= \sum_{i=1}^{n-1} a_i(x)\{xD^{i+1} - (n-i)D^i\}P_n^{(\alpha,\alpha)}(x) + \frac{4(2\alpha+3)_{n-2}}{(n-2)!}D^2P_n^{(\alpha,\alpha)}(x),$$

(3.9)
$$a_n(x) = \frac{-1}{D^nP_n^{(\alpha,\alpha)}(x)}\left[\sum_1^{n-1} a_i(x)D^iP_n^{(\alpha,\alpha)} + \frac{4(2\alpha+3)_{n-1}}{n!}D^2P_n^{(\alpha,\alpha)}(x)\right]$$

where the summation ranges over 1 through $n-1$, since $D^iP_n^{(\alpha,\alpha)}(x) = 0$ for $n < i$.
We note that (3.8) and (3.9) solve the equations (3.5) and (3.7) recursively in the following sense:

$\{a_i(x)\}_1^{n-1}$ determine $a_0(n,\alpha)$ and $a_n(x)$.
Again $\{a_i(x)\}_1^n$ determine $a_0(n+1,\alpha)$ and $a_{n+1}(x)$.

Substituting $P_n^{(\alpha,\alpha)}(x)$ into (3.8) and (3.9), we obtain the following expressions for $a_n(x)$ and $a_0(n,\alpha)$:

$$\begin{aligned}
&a_0(0,\alpha)=0,\\
&a_0(1,\alpha)=0,\quad a_1(x)=0,\\
&a_0(2,\alpha)=-4(2\alpha+3),\quad a_2(x)=2(2\alpha+3)(x^2-1),\\
&a_0(3,\alpha)=-4(2\alpha+3)(2\alpha+5),\quad a_3(x)=\frac{2}{3}(2\alpha+2)(2\alpha+3)(x^3-x),\\
&a_0(4,\alpha)=-2(2\alpha+3)(2\alpha+5)(2\alpha+6),\\
&a_4(x)=\frac{1}{6}(\alpha+1)(2\alpha+3)[(2\alpha+1)x^2-1](x^2-1),\\
&a_0(5,\alpha)=-\frac{2}{3}(2\alpha+3)(2\alpha+5)(2\alpha+6)(2\alpha+7),\\
&a_5(x)=\frac{1}{180}2\alpha(2\alpha+2)(2\alpha+3)x(x^2-1)[(2\alpha+1)x^2-3].
\end{aligned}$$

From the expressions for $a_n(x)$ and $a_0(n,\alpha)$ $(n\le 5)$ we conjecture that

$$a_0(n,\alpha)=\frac{-4(2\alpha+3)(2\alpha+5)_{n-2}}{(n-2)!}, \tag{3.10}$$

(3.11)

$$a_n(x)=\frac{4(2\alpha+3)}{n!}(x^2-1)\sum_{j=0}^{[\frac{n}{2}]-1}(-1)^j\binom{\alpha+1}{j}\binom{2\alpha+2-2j}{n-2-2j}x^{n-2j-2}.$$

CONJECTURE. *The generalized Jacobi polynomials* $\{P_n^{\alpha,\alpha,M,M}(x)\}_0^\infty$ *satisfy a differential equation of infinite order*

$$M\sum_0^\infty a_i(x)y^{(i)}(x)+(x^2-1)y''+(2\alpha+2)xy'-n(n+2\alpha+1)y=0. \tag{3.12}$$

where $\{a_i(x)\}_0^\infty$ *are given by (3.10) and (3.11).*

Moreover, the order of differential equation (3.12) is $(2\alpha+4)$ *if* α *is a nonnegative integer. Otherwise (3.12) is of infinite order.*

4. Generating Function

Suppose that α is a nonnegative integer. We know that the symmetric Jacobi polynomials $\{P_n^{(\alpha,\alpha)}(x)\}_0^\infty$ satisfy the following equation ([14])

$$\sum_0^\infty \binom{2\alpha}{\alpha}^{-1}\binom{n+2\alpha}{\alpha}P_n^{(\alpha,\alpha)}(x)\omega^n = (1-2x\omega+\omega^2)^{-\alpha-\frac{1}{2}}.$$

That is, $(1-2x\omega+\omega^2)^{-\alpha-\frac{1}{2}}$ is a generating function of $\{P_n^{(\alpha,\alpha)}(x)\}_0^\infty$. Using the formula (3.2), we have the following formula

$$\begin{aligned}
&\sum_0^\infty \binom{2\alpha}{\alpha}^{-1}\binom{n+2\alpha}{\alpha}\left\{1+M\frac{(2\alpha+2)_n n}{(\alpha+1)n!}\right\}^{-1} P_n^{\alpha,\alpha,M,M}(x)\omega^n \\
&=\sum_0^\infty \binom{2\alpha}{\alpha}^{-1}\binom{n+2\alpha}{\alpha}\left\{1+M\frac{(2\alpha+1)_n}{(2\alpha+1)n!}\right\} \\
&\quad\cdot\left\{-2x\frac{d}{dx}+\frac{n(n+2\alpha+1)}{\alpha+1}\right\}P_n^{(\alpha,\alpha)}(x) \\
&=\left[1+\frac{M}{(2\alpha+1)!}\left\{-2x\frac{\partial^{2\alpha+1}}{\partial x\partial\omega^{2\alpha}}\omega^{2\alpha}+\omega^2\frac{\partial^{2\alpha+2}}{\partial\omega^{2\alpha+2}}+\frac{3\alpha+1}{2\alpha+1}\omega\frac{\partial^{2\alpha+1}}{\partial\omega^{2\alpha+1}}\right\}\right]\cdot \\
&\quad\omega^{2\alpha}(1-2x\omega+\omega^2)^{-\alpha-\frac{1}{2}}
\end{aligned}$$

so that the right hand side is a generating function of $\{P_n^{\alpha,\alpha,M,M}(x)\}_0^\infty$.

5. Rodrigues' type formula

Note that

$$P_n^{(\alpha,\alpha)}(x) = \frac{(1-x^2)^{-\alpha}}{(-2)^n n!}D^n(1-x^2)^{n+\alpha}$$

and

$$x\frac{d}{dx}P_n^{(\alpha,\alpha)}(x) = nP_n^{(\alpha,\alpha)}(x) + \frac{n+\alpha}{n+2\alpha}\frac{d}{dx}P_{n-1}^{(\alpha\alpha)}(x).$$

From the formula of $\{P_n^{\alpha,\alpha,M,M}(x)\}_0^\infty$ (cf. (3.2)),

$$\begin{aligned}P_n^{\alpha,\alpha,M,M}(x) &= \left\{1+\frac{M(2\alpha+2)_n n}{(\alpha+1)n!}\right\}\left[\left\{1+\frac{M(2\alpha+3)_{n-2}}{(n-2)!}\right\}P_n^{(\alpha,\alpha)}(x)\right.\\ &\quad \left.-\frac{2M(n+\alpha)(2\alpha+2)_{n-1}}{n!(n+2\alpha)}\frac{d}{dx}P_n^{(\alpha,\alpha)}(x)\right]\\ &= A_n(1-x^2)^{-\alpha}D^n(1-x^2)^{n+\alpha}\\ &\quad + B_n(1-x^2)^{-\alpha}D^n(1-x^2)^{n-1+\alpha}\\ &\quad + C_n x(1-x^2)^{-\alpha-1}D^{n-1}(1-x^2)^{n-1+\alpha}\end{aligned}$$

where $A_n = \frac{1}{(-2)^n n!}\left\{1+\frac{M(2\alpha+2)_n n}{(\alpha+1)n!}\right\}\left\{1+\frac{M(2\alpha+3)_{n-2}n}{(\alpha-1)!}\right\}$,

$B_n = \frac{1}{(-2)^{n-1}(n-1)!}\left\{1+\frac{M(2\alpha+2)_n n}{(\alpha+1)n!}\right\}\frac{2M(n+\alpha)(2\alpha+2)_{n-1}}{n!(n+2\alpha)}$ and

$C_n = \frac{4M}{(-2)^{n-1}(n-1)!}\left\{1+\frac{M(2\alpha+2)_n n}{(\alpha+1)n!}\right\}\frac{\alpha(n+\alpha)(2\alpha+2)_{n-1}}{n!(n+2\alpha)}$.

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