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A NECESSARY AND SUFFICIENT CONDITION FOR $\mathbf{J(f,x_0,G)}$ TO BE ISOMORPHIC TO $\mathbf{J(f,x_0)\times G}$

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F.Rhodes [4] introduced the fundamental group $\sigma(X,x_0,G)$ of a transformation group (X,G) as a generalization of the fundamental group of a topological space X and showed a necessary condition for $\sigma(X,x_0,G)$ to be isomorphic to $\pi_1(X,x_0)\times G$, that is, if (G,G) admits a family of preferred paths at $e, \sigma(X,x_0,G)$ is isomorphic to $\pi_1(X,x_0)\times G$. B.J. Jiang [3] introduced the Jiang subgroup $J(f,x_0)$ of the fundamental group of a topological space X. The authors [8] introduced the extended Jiang subgroup $J(f,x_0,G)$ of the fundamental group of a transformation group as a generalization of the Jiang subgroup $J(f,x_0)$.

In this paper, we give a necessary and sufficient condition for $J(f,x_0,G)$ to be isomorphic to $J(f,x_0)\times G$.

Let (X,G,π) be a transformation group, where X is a path with connected space with x_0 as base point. Given any element g of G, a path f of order g with base point x_0 is a continuous map $f: I \longrightarrow X$ such that $f(0)=x_0$ and $f(1)=gx_0$. A path f_1 of order g_1 and a path f_2 of order g_2 give rise to a path $f_1+g_1f_2$ of order g_1g_2 defined by the equations

$$(f_1+g_1f_2)(s)=\begin{cases} f_1(2s), & 0\le s\le 1/2 \\ g_1f_2(2s), & 1/2\le s\le 1. \end{cases}$$

Two paths f and f' of the same order g are said to be homotophic if there is a continuous map $F: I^2 \longrightarrow X$ such that

$$\begin{aligned} &F(s,0)=f(s),\ 0\le s\le 1,\\ &F(x,1)=f'(s),\ 0\le s\le 1,\\ &F(0,t)=x_0,\ 0\le t\le 1,\\ &F(1,t)=gx_0,\ 0\le t\le 1. \end{aligned}$$

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The homotopy class of a path f of order g is denoted by $[f : g]$. Two homotopy classes of paths of different orders g_1 and g_2 are distinct, even if $g_1x_0 = g_2x_0$. F.Rhodes showed that the set of homotopy classes of paths of prescribed order with the rule of composition $*$ is a group, where $*$ is defined by $[f_1 : g_1] * [f_2 : g_2] = [f_1 + g_1f_2 : g_1g_2]$. This group is denoted by $\sigma(X, x_0, G)$ and is called the *fundamental group* of (X, G) with base point x_0.

Let f be a self-map of X. A homotopy $H : X \times I \longrightarrow X$ is called a *cyclic* homotopy if $H(x, 0) = H(x, 1) = f(x)$. In [3], this concept of a topological space was generalized as follows : A continuous map $H : X \times I \longrightarrow X$ is called an f-homotopy of order g if $H(x, 0) = f(x), H(x, 1) = gf(x)$, where g is an element of G. In [8], an extended Jiang subgroup $J(f, x_0, G)$ was defined by $J(f, x_0, G) = \{[\alpha : g] \in \sigma(X, f(x_0), G) |$there exists an f-homotopy of order g with trace $\alpha\}$. In particular, the Jiang subgroup $J(f, x_0)$ [3] can be identified by $J(f, x_0, \{e\})$.

In [4], a transformation group (X, G) is said to admit a family of preferred paths at x_0 if it is possible to associate with every element g of G a path k_g from gx_0 to x_0 such that the path k_e associated with the identity element e of G is $\hat{x}_0$ which is the constant map such that $\hat{x}_0(t) = x_0$ for each $t \in I$ and for every pair of elements g, h the path k_{gh} from ghx_0 to x_0 is homotopic to $gk_h + k_g$.

DEFINITION 1. A family K of preferred paths at $f(x_0)$ is called a family of preferred f-traces at x_0 if for every preferred path k_g in K, $k_g\rho$ is the trace of f-homotopy of order g.

THEOREM 2. *Let (X, G, π) be a transformation group. If (G, G) admits a family of preferred paths at e, then (X, G) admits a family of preferred f-traces at x_0 for any self map f of X.*

Proof. Let H be a family of preferred paths at e in (G, G). Define $K = \{k_g | k_g(t) = h_g(t)(f(x_0)), h_g \in H\}$. Let $F : X \times I \longrightarrow X$ be the map such that

$$F(x, t) = \pi(f(x), h_g\rho(t)), \rho(t) = 1 - t.$$

So,

$$F(x, 0) = \pi(f(x), h_g(1)) = h_g(1)f(x) = f(x),$$
$$F(x, 1) = \pi(f(x), h_g(0)) = h_g(0)f(x) = gf(x)$$

and

$$F(x_0, t) = \pi(f(x_0), h_g\rho(t)) = h_g\rho(t)f(x_0) = k_g\rho(t).$$

Thus, F is a f-homotopy of order g with trace $k_g\rho$. So, K is a family of preferred f-traces at x_0.

LEMMA 3. *Let (X, G) be a transformation group and let $f : X \longrightarrow X$ be a self map. If k is a trace of a f-homotopy of order g, then for every loop α at x_0, $f\alpha$ is homotopic to $k + gf\alpha + k\rho$. In particular, if f is a homeomorphism and α is a loop at $f(x_0)$, α is homotopic to $k + g\alpha + k\rho$.*

Proof. Let $H : X \times I \longrightarrow X$ be a f-homotopy of order g with trace k and α be a loop at x_0. Define $F : I \times I \longrightarrow X$ by

$$F(x,t) = \begin{cases} k(4s), & 0 \le s \le t/4 \\ H(\alpha((4s-t)/(4-2t)), t), & t/4 \le s \le (4-t)/4 \\ k\rho(4s-3), & (4-t)/4 \le s \le 1. \end{cases}$$

Then F is well defined and

$$F(s,0) = H(\alpha(s), 0) = (f\alpha)(s),$$
$$F(x,1) = (k + gf\alpha + k\rho)(s).$$

In particular, suppose that f is a homeomorphism. Define $F : X \times I \longrightarrow X$ by

$$F(x,t) = \begin{cases} k(4s), & 0 \le s \le t/4 \\ H(f^{-1}\alpha((4s-t)/(4-2t)), t), & t/4 \le s \le (4-t)/4 \\ k\rho(4s-3), & (4-t)/4 \le s \le 1. \end{cases}$$

Then $F(s,0) = H(f^{-1}\alpha(s),0) = f(f^{-1}\alpha(s)) = \alpha(s)$.

$$
\begin{aligned}
F(s,1) &= \begin{cases} k(4s), & 0 \le s \le 1/4 \\ H(f^{-1}(\alpha((4s-1)/2)),1), & 1/4 \le s \le 3/4 \\ k\rho(4s-3), & 3/4 \le s \le 1, \end{cases} \\
&= \begin{cases} k(4s), & 0 \le s \le 1/4 \\ gff^{-1}(\alpha((4s-1)/2)), & 1/4 \le s \le 3/4 \\ k\rho(4s-3), & 3/4 \le s \le 1, \end{cases} \\
&= \begin{cases} k(4s), & 0 \le s \le 1/4 \\ g\alpha((4s-1)/2), & 1/4 \le s \le 3/4 \\ k\rho(4s-3), & 3/4 \le s \le 1, \end{cases} \\
&= (k + g\alpha + k\rho)(s).
\end{aligned}
$$

Therefore α is homotopic to $k + g\alpha + k\rho$.

THEOREM 4. *A transformation group (X,G) admits a family of preferred f-traces at x_0 if and only if $J(f,x_0,G)$ is a split extension of $J(f,x_0)$ by G.*

Proof. Suppose (X,G) admits a family $K = \{k_g | g \in G\}$ of preferred f-traces at x_0. Consider the sequence:

$$
O \longrightarrow J(f,x_0) \xrightarrow{\ i_g\ } J(f,x_0,G) \xrightarrow{\ i_G\ } G \longrightarrow O,
$$

where $i_G([\alpha]) = [\alpha : e]$ and $j_G[\alpha : g] = g$. Since i_G is a monomorphism, j_G is an epimorphism and Ker j_G = Im i_G, the sequence is a short exact sequence. Define $\psi : G \longrightarrow J(f,x_0,G)$ by $\psi(g) = [k_g\rho : g]$. Then ψ is a homomorphism. Indeed,

$$
\begin{aligned}
\psi(g_1g_2) &= [k_{g_1g_2}\rho : g_1g_2] \\
&= [(g_1k_{g_2} + k_{g_1})\rho : g_1g_2] \\
&= [k_{g_1}\rho + g_1k_{g_2}\rho : g_1g_2] \\
&= [k_{g_1}\rho : g_1] * [k_{g_2}\rho : g_2] \\
&= \psi(g_1) * \psi(g_2).
\end{aligned}
$$

By definition of ψ, we have $j_G \circ \psi = 1_G$. Thus $J(f, x_0, G)$ is a split extension of $J(f, x_0)$ by G.

Conversely, suppose $J(f, x_0, G)$ is a split extension of $J(f, x_0)$ by G. Then there is a monomorphism $\psi : G \longrightarrow J(f, x_0, G)$ such that $j_G \circ \psi = 1_G$. Let $H = \{\alpha_g | \alpha_g \rho$ is a representative path of $\psi(g)\}$. Since $\psi(e) = [\hat{f}(x_0) : e]$ and $\psi(g_1 g_2) = \psi(g_1) * \psi(g_2)$, α_g is a path from $gf(x_0)$ to $f(x_0)$ for each element g of G , $\alpha_e = \hat{f}(x_0)$ and $\alpha_{g_1 g_2}$ is homotopic to $g_1 \alpha_{g_2} + \alpha_{g_1}$. So, H is a family of preferred f-traces at x_0. Therefore, a transformation group (X, G) admits a family of preferred f-traces at x_0.

THEOREM 5. *Let $f : X \longrightarrow X$ be a homeomorphism. A transfromation group (X, G) admits a family of preferred f-traces at x_0 if and only if there esixts an isomorphism $\phi : J(f, x_0, G) \longrightarrow J(f, x_0) \times G$ such that the diagram commmutes*

$$\begin{array}{ccccccccc} & & & & J(f,x_0,G) & & & & \\ & & & \nearrow & & \searrow & & & \\ 0 & \rightarrow & J(f,x_0) & & \downarrow \phi & & G & \rightarrow & 0. \\ & & & \searrow & & \nearrow & & & \\ & & & & J(f,x_0)\times G & & & & \end{array}$$

Proof. Let $K = \{k_g | g \in G\}$ be a family of preferred f-trace at x_0. Define $\phi : J(f, x_0, G) \longrightarrow J(f, x_0) \times G$ by $\phi([\alpha : g]) = ([\alpha + k_g], g)$. Let $[\alpha : g]$ be an element of $J(f, x_0, G)$. Then there exists a f-homotopy $H : X \times I \longrightarrow X$ of order g such that $H(x, 0) = f(x), H(x, 1) = gf(x)$ and $H(x_0, t) = \alpha(t)$, and $k_g \rho$ is a trace of f-homotopy $J : X \times I \longrightarrow X$ of order g.

Define $F : X \times I \longrightarrow X$ by

$$F(x,t) = \begin{cases} H(x, 2t), & 0 \le t \le 1/2 \\ J(x, 2(1-t)), & 1/2 \le t \le 1. \end{cases}$$

Then F is a cyclic homotopy with trace $\alpha + k_g$, for

$$F(x, 0) = H(x, 0) = f(x), F(x, 1) = J(x, 0) = f(x),$$

$$\begin{aligned} F(x_0, t) &= \begin{cases} H(x_0, t), & 0 \le t \le 1/2 \\ J(x_0, 2(1-t)), & 1/2 \le t \le 1 \end{cases} \\ &= (\alpha + k_g)(t). \end{aligned}$$

Thus $[\alpha + k_g]$ belongs to $J(f, x_0)$. Let $[\alpha : g] = [\alpha' : g']$. Then α is homotopic to $\alpha', g = g'$ and $\alpha + k_g$ is also homotopic to $\alpha' + k_g$. Thus ϕ is well-defined. Suppose $\phi([\alpha : g]) = \phi([\alpha' : g])$. Then $\alpha + k_g$ is homotopoic to $\alpha' + k_g$. This implies that $\alpha (= \alpha + k_g + k_g\rho)$ is homotopic to $\alpha' (= \alpha' + k_g + k_g\rho)$. Therefore ϕ is injective.

For any element $([\alpha], g) \in J(f, x_0) \times G$, there exists a cyclic homotopy $H : X \times I \longrightarrow X$ such that $H(x, 0) = f(x) = H(x, 1)$ and $H(x_0, t) = \alpha(t)$. Since $\{k_g | g \in G\}$ is a family of preferred f-traces at x_0, there exists a f-homotopy $W : X \times I \longrightarrow X$ such that $W(x, 0) = f(x), W(x, 1) = gf(x)$ and $W(x_0, t) = k_g\rho(t)$. Define

$$F(x,t) = \begin{cases} H(x, 2t), & 0 \le t \le 1/2 \\ W(x, 2t-1), & 1/2 \le t \le 1, \end{cases}$$

then $F(x_0, t) = (\alpha + k_g\rho)(t)$. So, there exists an element $([\alpha + k_g\rho + k_g], g) = ([\alpha], g)$. Therefore, ϕ is surjective.

Next, we show that ϕ is a homomorphism. Let $[\alpha_1 : g_1]$ and $[\alpha_2 : g_2]$ be elements of $J(f, x_0, G)$. Then

$$\begin{aligned} \phi([\alpha_1 : g_1] * [\alpha_2 : g_2]) &= \phi([\alpha_1 + g_1\alpha_2 : g_1 g_2]) \\ &= ([\alpha_1 + g_1\alpha_2 + k_{g_1 g_2}], g_1 g_2), \end{aligned}$$

while

$$\begin{aligned} \phi([\alpha_1 : g_1]) \circ \phi([\alpha_2 : g_2]) &= ([\alpha_1 + k_{g_1}], g_1) \circ ([\alpha_2 + k_{g_2}], g_2) \\ &= ([\alpha_1 + k_{g_1} + \alpha_2 + k_{g_2}], g_1 g_2). \end{aligned}$$

Since $\alpha_2 + kg_2$ is a loop at $f(x_0)$ and $k_{g_1}\rho$ is a trace of a f-homotopy of order $g_1, \alpha_2 + k_{g_2}$ is homotopic to $k_{g_1}\rho + g_1(\alpha_2 + k_{g_2}) + k_{g_1}$ by Lemma 3. Therefore, we have

$$\begin{aligned} &\alpha_1 + k_{g_1} + \alpha_2 + k_{g_2} \sim \alpha_1 + k_{g_1} + k_{g_1}\rho + g_1(\alpha_2 + k_{g_2}) \\ &+k_{g_1} \sim \alpha_1 + g_1(\alpha_2 + k_{g_2}) + k_{g_1} \sim \alpha_1 + g_1\alpha_2 + g_1 k_{g_2} \\ &+k_{g_1} \sim \alpha_1 + g_1\alpha_2 + k_{g_1 g_2}. \end{aligned}$$

This implies that ϕ is a homomorphism.

Conversely, given a commutative diagram with exact rows and ϕ an isomorphism:

$$\begin{array}{ccccccccc}
 & & & & J(f,x_0,G) & & & & \\
 & & & i_G \nearrow & & j_G \searrow & & & \\
0 & \rightarrow & J(f,x_0) & & \downarrow \phi & & G & \rightarrow & 0, \\
 & & & i_1 \searrow & & \pi_2 \nearrow \swarrow i_2 & & & \\
 & & & & J(f,x_0) \times G & & & &
\end{array}$$

define $\psi : G \longrightarrow J(f, x_o, G)$ to be $\phi^{-1} \circ i_2$. Use the commutativity of the diagram to show $j_G \circ \psi = 1_G$. Then $J(f, x_0, G)$ is a split extension of J(f, x_0) by G. By Theorem 4, (X, G) admits a family of preferred f-traces at x_0.

COROLLARY 6. *Let $f : X \longrightarrow X$ be a homeomorphism. A transformation group (X, G) admits a family of preferred f-traces at x_0 and G abelian if and only if $O \longrightarrow J(f, x_0) \longrightarrow J(f, x_0, G) \longrightarrow G \longrightarrow O$ is a split exact sequence of Z-module.*

We show that the existence of a family of preferred f-traces on a transformation group does not depend on base point.

THEOREM 7. *Let (X, G) be a transformation group. If λ is a path from x_0 to x_1, then a family of preferred f-traces at x_0 gives rise to a family of preferred f-traces at x_1.*

Proof. Let $K = \{k_g | g \in G\}$ be a family of preferred f-traces at x_0. For each element g of G, let $h_g = gf\lambda\rho + k_g + f\lambda$. Then $H = \{h_g | g \in G\}$ is a family of preferred f-traces at x_1. Because, $h_e = f\lambda\rho + k_e + f\lambda \sim \hat{f}(x_1)$ and

$$\begin{aligned}
h_{g_1 g_2} &= (g_1 g_2) f\lambda\rho + k_{g_1 g_2} + f\lambda \\
&\sim (g_1 g_2) f\lambda\rho + g_1 k_{g_2} + k_{g_1} + f\lambda \\
&\sim (g_1 g_2) f\lambda\rho + g_1 k_{g_2} + g_1 f\lambda + g_1 f\lambda\rho + k_{g_1} + f\lambda \\
&\sim g_1(g_2 f\lambda\rho + k_{g_2} + f\lambda) + (g_1 f\lambda\rho + k_{g_1} + f\lambda) \\
&\sim g_1 h_{g_2} + h_{g_1}.
\end{aligned}$$

Since the induced isomorphism $(f\lambda)_*$ carries $J(f, x_0, G)$ isomorphically onto $J(f, x_1, G)$ by Theorem 8 in [8], $(f\lambda)_*[k_g\rho : g] = [f\lambda\rho + k_g\rho + gf\lambda : g] = [h_g\rho : g]$ belongs to $J(f, x_1, G)$ for any element $[k_g\rho : g]$ of $J(f, x_0, G)$. Thus $H = \{h_g | g \in G\}$ is a family of preferred f-traces at x_1.

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