Comm. Korean Math. Soc. **8** (1993), No. 1, pp. 39–45

PRIME SPECTRA OF FINITELY GENERATED MODULES

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In this paper, *unless otherwise indicated, we shall not assume that our rings are commutative, but we shall always assume that every ring has an identity element.* By *a module*, we shall always mean a unitary left module.

DEFINITION. Let E be an R-module. Then a submodule A of E is called *an (resp. prime, radical) extended submodule* if there exists an (resp. prime, radical) ideal $\mathfrak{a}$ in R such that $A = \mathfrak{a}E$.

LEMMA 1. *Let E be a finitely generated R-module. Then every extended submodule of E is of the form $\mathfrak{a}E$, where $\mathfrak{a}$ is an ideal of R containing $\operatorname{Ann}_R E$.*

Proof. This follows immediately from the fact that for every ideal I of R, $IE = (I + \operatorname{Ann}_R E)E$.

DEFINITION. If $\mathfrak{a}$ is an ideal in a ring R and n is a positive integer, the *n-th radical* of $\mathfrak{a}$ in R is defined by

$$\sqrt[n]{\mathfrak{a}} = \{x \in R \mid x^n \in \mathfrak{a}\}.$$

$\sqrt[1]{\mathfrak{a}} = \mathfrak{a}$. $\sqrt[n]{\mathfrak{a}}$ is merely a subset of R contained in the radical $\sqrt{\mathfrak{a}}$ of $\mathfrak{a}$ in R. However, if R is a commutative ring and is of prime characteristic p, then $\sqrt[p]{\mathfrak{a}}$ forms an ideal of R.

LEMMA 2. *Let R be a commutative ring and E an R-module generated by n elements. If $\mathfrak{a}$ is an ideal in R containing $\operatorname{Ann}_R E$, then*

$$\mathfrak{a} \subseteq \operatorname{Ann}_R(E/\mathfrak{a}E) \subseteq \sqrt[n]{\mathfrak{a}} \subseteq \sqrt{\mathfrak{a}}.$$

(Of course, if $n = 1$, the commutativity condition on R can be omitted.)

Received March 22, 1992.

In particular, if either E is cyclic or $\mathfrak{a}$ is a radical ideal in R containing $\operatorname{Ann}_R E$, *then*

$$\operatorname{Ann}_R(E/\mathfrak{a}E) = \mathfrak{a}.$$

Moreover, for every proper ideal $\mathfrak{a}$ in R containing $\operatorname{Ann}_R E$, $\mathfrak{a}E \neq E$.

Proof. Use Theorem 75 of [K70].

THEOREM 3. *Let E be a cyclic R-module. E satisfies the ascending chain condition (resp. descending chain condition) on extended submodules if and only if the residue class ring $R/\operatorname{Ann}_R E$ is Noetherian (resp. Artinian).*

Proof. The *if* part follows from Lemmas 1, 2, and the *only if* part follows from Lemma 2.

THEOREM 4. *Let R be a commutative ring and E a finitely generated R-module which satisfies the ascending chain condition on prime (resp. radical) extended submodules. Then the residue class ring $R/\operatorname{Ann}_R E$ satisfies the ascending chain condition on prime (resp. radical) ideals.*

Moreover, this statement also holds in the case that "the ascending chain condition" is replaced by "the descending chain condition".

Proof. This follows from the definitions and Lemma 2.

We recall the definition [L91, p.1] : in an R-module E, a submodule P is called *a prime submodule* of E if (a) P is proper and (b) whenever $re \in P$ ($r \in R$, $e \in E$), then either $e \in P$ or $rE \subseteq P$. This definition is the natural generalization of the one of a prime ideal in a commutative ring.

Let E be a non-zero finitely generated R-module. There is no guarantee that $\mathfrak{p}E$ is a prime R-submodule in E, even if $\mathfrak{p}$ is a prime ideal in R containing $\operatorname{Ann}_R E$. We will discuss under what conditions $\mathfrak{p}E$ is a prime R-submodule in E provided that $\mathfrak{p}$ is a prime ideal of R containing $\operatorname{Ann}_R E$.

PROPOSITION 5. *Let R be a commutative ring and E a free R-module with a finite basis. Then every prime extended submodule in E is prime.*

LEMMA 6. *Let R be a commutative quasi-local ring, and E a finitely generated R-module. Then E is quasi-local if and only if E is cyclic.*

Proof. See Corollary to Theorem 13 of [L91].

LEMMA 7. *Every quasi-local finitely generated R-module E is indecomposable.*

Proof. If E is decomposable there exist submodules A, B of E such that $E = A \oplus B$, and $A \neq 0$, $B \neq 0$. But then $A \neq E \neq B$. Let M be the unique maximal submodule of E. Then $A + B \subseteq M \neq E$, which contradicts.

LEMMA 8. *The injective envelope of a submodule, within a given non-singular module, is unique.*

Proof. Assume that E and E' are both injective envelopes of a submodule N, within a given non-singular R-module M. Then it is well-known [G76, Proposition 1.11] that there exists an isomorphism f from E onto E' such that $f|_N = \mathrm{id}_N$.

E is non-singular because it is a submodule of the non-singular module M. Since E is an essential extension of N, it follows from [G76, Proposition 1.21] that E/N is singular. We now consider the following diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & {}_{\mathrm{inc}}\searrow \quad \swarrow_{\mathrm{inc}} & \\ & M & \end{array}$$

The two homomorphisms $E \xrightarrow{\mathrm{inc}} M$, $E \xrightarrow{f} E' \xrightarrow{\mathrm{inc}} M$ agree on N. Therefore by [G76, Lemma 2.1] they must be equal and hence $E = f(E) = E'$ in M.

LEMMA 9. *Let $M = \sum_{i \in I} E_i$ be a non-singular R-module which is a sum of indecomposable injective submodules E_i. Then there exists a subset J of I such that $M = \sum_{j \in J} E_j(d.s.)$.*

Proof. Consider the family $\{E_i\}_{i \in I}$. Then by [SV72, Proposition 1.7] there is a maximal subfamily, say $\{E_j\}_{j \in J}$, of $\{E_i\}_{i \in I}$ such that the sum $\sum_{j \in J} E_j$ is direct. Let $C = \sum_{j \in J} E_j(d.s.)$. It now suffices to prove that $C = M$.

Suppose on the contrary that $C \neq M$. Then there exists i in I such that $E_i \not\subseteq C$. By the maximality of $\{E_j\}_{j\in J}$, $E_i \cap C \neq 0$. This implies that there exists a finite number of elements $j_1, j_2, \ldots, j_n$ in J such that $E_i \cap (E_{j_1} \oplus E_{j_2} \oplus \cdots \oplus E_{j_n}) \neq 0$. We put

$$P = E_{j_1} \oplus E_{j_2} \oplus \cdots \oplus E_{j_n},$$
$$Q = E_i \cap P,$$

so that $Q \neq 0$. Since P is injective, Q has an injective envelope $E(Q)$ which is a submodule of P [SV72, Proposition 2.22]. Further, E_i is an injective envelope $E'(Q)$ of Q [SV72, Proposition 2.28]. Therefore by Lemma 8, $E_i = E'(Q) = E(Q) \subseteq P \subseteq C$. This gives a contradiction and shows that $C = M$.

Let E be an R-module which is a direct sum of a finite number of cyclic submodules, say $Rd_1, \ldots, Rd_m$. If $\mathfrak{p}$ is a prime ideal of R containing $(0 : d_1) + (0 : d_2) + \cdots + (0 : d_m)$, then $\mathfrak{p}E$ is a prime R-submodule of E. Hence, from Lemmas 6, 7, 9, we get the following result.

THEOREM 10. *Let R be a commutative quasi-local ring and $E = Re_1 + Re_2 + \cdots + Re_n$ a non-singular finitely generated R-module with the property that each Re_i is an injective R-submodule. If $\mathfrak{p}$ is a prime ideal of R containing $(0 : e_1) + (0 : e_2) + \cdots + (0 : e_n)$ then $\mathfrak{p}E$ is a prime R-submodule of E.*

Let E be a non-zero finitely generated R-module. Then the prime spectrum of E, denoted by $\operatorname{Spec}_R E$, is defined to be the collection of all the prime R-submodules in E. Let $\bar{R}$ denote $R/\operatorname{Ann}_R E$. Then the mapping $f : \operatorname{Spec}_R E \to \operatorname{Spec}_R \bar{R}$ defined by $f(P) = \overline{\operatorname{Ann}_R(E/P)}$, where $P \in \operatorname{Spec}_R E$, is neither surjective nor injective, in general. If E is a free module with a finite basis over a commutative ring R, then the mapping $f : \operatorname{Spec}_R E \to \operatorname{Spec}_R R$ defined as above, is surjective (Proposition 5 and Lemma 2), but not injective, in general.

Let R be a commutative ring and E a free R-module with a finite basis. We now restrict our attention to the *minimal prime spectrum* of E, denoted by $\operatorname{Min}_R E$, which is defined to be the collection of all the minimal prime R-submodules in E. The image of its restriction $f|_{\operatorname{Min}_R E} : \operatorname{Min}_R E \to \operatorname{Spec}_R R$ to the minimal prime spectrum $\operatorname{Min}_R E$

is $\mathrm{Min}_R R$. In fact, let P be a minimal prime submodule in E. Consider a prime ideal $\mathfrak{q}$ in R satisfying $\mathfrak{q} \subseteq \mathrm{Ann}_R(E/P)$. Then $\mathfrak{q}E \subseteq P$. But $\mathfrak{q}E$ is a prime submodule in E and $\mathrm{Ann}_R(E/\mathfrak{q}E) = \mathfrak{q}$. Since P is minimal, we must have $\mathfrak{q}E = P$ and $\mathfrak{q} = \mathrm{Ann}_R(E/P)$. Hence $\mathrm{Ann}_R(E/P)$ is a minimal prime ideal in R.

Conversely, if $\mathfrak{p}$ is a minimal prime ideal in R, then $\mathfrak{p}E$ is a prime submodule in E. Further, let Q be a prime submodule in E and $Q \subseteq \mathfrak{p}E$. Then $\mathrm{Ann}_R(E/Q) \subseteq \mathrm{Ann}_R(E/\mathfrak{p}E) = \mathfrak{p}$, so $\mathfrak{p} = \mathrm{Ann}_R(E/Q)$. Hence $\mathfrak{p}E \subseteq Q$, and therefore $Q = \mathfrak{p}E$. Consequently, $\mathfrak{p}E$ is a minimal prime submodule and $\mathrm{Ann}_R(E/\mathfrak{p}E) = \mathfrak{p}$.

By the above argument, the mapping $f|_{\mathrm{Min}_R E} : \mathrm{Min}_R E \to \mathrm{Min}_R R$ is surjective. Moreover, it is injective. In fact, assume that P and Q are minimal prime submodules in E satisfying $\mathrm{Ann}_R(E/P) = \mathrm{Ann}_R(E/Q)$. Then $(\mathrm{Ann}_R(E/P))E \subseteq P$, so $P = (\mathrm{Ann}_R(E/P))E$. Similarly, $Q = (\mathrm{Ann}_R(E/Q))E$. Hence, $P = Q$.

Let us summarize the results as follows :

THEOREM 11. *Let R be a commutative ring and E a non-zero free R-module with a finite basis. Then there is a one-to-one order-preserving correspondence between all the minimal prime R-submodules in E and all the minimal prime ideals in R, given alternatively by $P \leftrightarrow \mathrm{Ann}_R(E/P)$ or $\mathfrak{p}E \leftrightarrow \mathfrak{p}$.*

COROLLARY. *Let R be a commutative ring and E a non-zero free R-module with a finite basis. Then every minimal prime submodule in E is of the form $\mathfrak{p}E$, where $\mathfrak{p}$ is a minimal prime ideal in R containing $\mathrm{Ann}_R E$.*

Let R be a commutative ring and E a non-zero free R-module with a finite basis. Then, it follows from Lemma 2 and Theorem 11 that

$$\mathrm{Card}(\mathrm{Min}_R E) = \mathrm{Card}(\mathrm{Min}_R R) \leq \mathrm{Card}(\mathrm{Spec}_R R) \leq \mathrm{Card}(\mathrm{Spec}_R E),$$

where $\mathrm{Card}(A)$ means the cardinality of a set A.

If a module is non-singular, then so is every module which is isomorphic to it.

PROPOSITION 12. *If E is a non-singular R-module, then so is $R/\operatorname{Ann}_R E$.*

Proof. Assume that E is a non-singular R-module. Set $\bar{R} = R/\operatorname{Ann}_R E$ and let $\operatorname{Ann}_R \bar{x}$ be an essential ideal in R, where $\bar{x}$ denotes the residue class of $x \in R$ modulo $\operatorname{Ann}_R E$. Then $\operatorname{Ann}_R(xE)$ is also an essential ideal in R. In fact, for any non-zero ideal I in R, $I \cap \operatorname{Ann}_R \bar{x} \neq 0$. Take $i \in I$ with $i \neq 0$ and $i\bar{x} = \bar{0}$ in $\bar{R}$. Then $ix \equiv 0 \bmod (\operatorname{Ann}_R E)$, hence $i(xE) = 0$ in E. This implies that $I \cap \operatorname{Ann}_R(xE) \neq 0$.

By our assumption, $xE = 0$ in E. Hence $\bar{x} = \bar{0}$ in $\bar{R}$.

LEMMA 13. *If E is a non-singular module over a commutative ring R, then the ideal $\operatorname{Ann}_R E$ of R is radical.*

Proof. Assume that E is a non-singular module over a commutative ring R. Let $\bar{R}$ denote the residue class ring $R/\operatorname{Ann}_R E$. It suffices to prove that $\bar{R}$ is reduced.

Let $\bar{x}$ be any nilpotent element in $\bar{R}$, where $\bar{x}$ denotes the residue class of $x \in R$ modulo $\operatorname{Ann}_R E$. Then it follows from [K70, §1-3, Exercise 14, p.21] that $\bar{x}$ is a zero-divisor on every ideal in $\bar{R}$. Hence $\operatorname{Ann}_{\bar{R}}(\bar{x})$ is an essential ideal in $\bar{R}$, so that $I \cap \operatorname{Ann}_R(xE) \neq 0$ for any ideal I of R not contained in $\operatorname{Ann}_R(E)$. Further, $xE \subseteq E$ implies $\operatorname{Ann}_R(E) \subseteq \operatorname{Ann}_R(xE)$, and hence $I \cap \operatorname{Ann}_R(xE) = I$ for any ideal I of R contained in $\operatorname{Ann}_R E$. Thus, the intersection of any non-zero ideal I in R and $\operatorname{Ann}_R(xE)$ is non-zero. It follows that $\operatorname{Ann}_R(xE)$ is an essential ideal in R. Therefore, by our assumption, $xE = 0$, and hence $\bar{x} = \bar{0}$ in $\bar{R}$.

COROLLARY 1. *If E is a non-singular faithful module over a commutative ring R, then R is reduced.*

COROLLARY 2. *Every non-singular commutative ring is reduced.*

THEOREM 14. *Let E be a non-zero cyclic module over a commutative ring R and P a minimal prime R-submodule in E. Let $\mathfrak{p}$ denote $\operatorname{Ann}_R(E/P)$. If E is non-singular, then $E_{\mathfrak{p}}$ is a simple $R_{\mathfrak{p}}$-module.*

Proof. $E_{\mathfrak{p}}$ is a quasi-local $R_{\mathfrak{p}}$-module with unique maximal submodule $(\mathfrak{p}R_{\mathfrak{p}})E_{\mathfrak{p}}$ [L91, Theorem 13]. To show that $E_{\mathfrak{p}}$ is simple, it suffices to prove that $(\mathfrak{p}R_{\mathfrak{p}})E_{\mathfrak{p}} = 0$, i.e. $\mathfrak{p}R_{\mathfrak{p}} = \operatorname{Ann}_{R_{\mathfrak{p}}} E_{\mathfrak{p}}$.

Since P is a minimal prime submodule of E, it follows from the statement just prior to Theorem 10, and Lemma 2 that $\mathfrak{p}$ is a minimal prime ideal in R containing $\operatorname{Ann}_R E$. The connection between the prime ideals in R and those in $R_\mathfrak{p}$ shows that $\mathfrak{p}R_\mathfrak{p}$ is the only prime ideal in $R_\mathfrak{p}$ containing $(\operatorname{Ann}_R E)R_\mathfrak{p}$. It is well-known [N76, Chapter 2, Theorem 12, p.41] that $(\operatorname{Ann}_R E)R_\mathfrak{p} = \operatorname{Ann}_{R_\mathfrak{p}} E_\mathfrak{p}$. Hence $\sqrt{\operatorname{Ann}_{R_\mathfrak{p}} E_\mathfrak{p}} = \mathfrak{p}R_\mathfrak{p}$ [N76, Chapter 4, Theorem 10, p.113]. Now, it suffices to show that $\operatorname{Ann}_{R_\mathfrak{p}} E_\mathfrak{p}$ is a radical ideal in $R_\mathfrak{p}$.

By our assumption and Lemma 13, $\operatorname{Ann}_R E$ is a radical ideal in R. Further, we know, from [AM69, Proposition 3.11, v), p.42], that $\sqrt{(\operatorname{Ann}_R E)R_\mathfrak{p}} = (\sqrt{\operatorname{Ann}_R E})R_\mathfrak{p}$. Therefore, $\operatorname{Ann}_{R_\mathfrak{p}} E_\mathfrak{p}$ is a radical ideal in $R_\mathfrak{p}$.

COROLLARY. *If $\mathfrak{p}$ is a minimal prime ideal in a non-singular commutative ring R, then $R_\mathfrak{p}$ is a field.*

Of course, this corollary can be proved by using Corollary 2 to Lemma 13. In fact, if $\mathfrak{p}$ is a minimal prime ideal in a commutative reduced ring R, then $R_\mathfrak{p}$ is a field.

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