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NOTES ON MEDIAL BCI–ALGEBRAS

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In this note, we give a simpler axiomarization of medial BCI-algebras and some interesting corollaries.

An algebra $< X; *, 0 >$ of type (2, 0) is said to be a BCI-algebra if it satisfies:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A BCI-algebra X has the following properties:
(1) $x * 0 = x$.
(2) $(x * y) * z = (x * z) * y$.

The notion of medial BCI-algebras was introduced by W. A. Dudek [3]. In [2], M. A. Chaudhary and B. Ahmad discussed such an algebra.

DEFINITION 1. A BCI-algebra $< X; *, 0 >$ is said to be medial if it satisfies

(3) $(x * y) * (z * u) = (x * z) * (y * u)$
for all x, y, z and u of X.

In a medial BCI-algebra the following holds:
(4) $x * (x * y) = y$.

Now we give a simpler axiomarization of such an algebra, which is the main result of this note.

THEOREM 2. *An algebra* $< X; *, 0 >$ *of type (2, 0) is a medial BCI-algebra if and only if it satisfies*

(5) $(x * 0) * (y * z) = z * (y * x)$,

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(6) $x * (y * y) = x$.

Proof. ($\Leftarrow$) By (6) and (5), we have

$$x * 0 = (x * 0) * (x * x) = x * (x * x) = x.$$

(1) holds. Let $y = x$ and $z = 0$ in (5), and using (1) and (6) we obtain

$$x * x = (x * 0) * (x * 0) = 0 * (x * x) = 0.$$

(III) holds. By (1) and (5), we have

$$x * (y * z) = z * (y * x). \tag{7}$$

Combining (7) and (6), we obtain

$$x * (x * y) = y * (x * x) = y.$$

(4) is true. As

$$\begin{aligned}((x * y) * (x * z)) * (z * y) &= (z * (x * (x * y))) * (z * y) \qquad [\text{by } (7)]\\ &= (z * y) * (z * y) \qquad [\text{by } (4)]\\ &= 0, \qquad [\text{by } (\text{III})]\end{aligned}$$

(I) holds. By (4) and (III),

$$(x * (x * y)) * y = y * y = 0.$$

(II) holds. Suppose $x * y = y * x = 0$. By (1) and (4) we have

$$x = x * 0 = x * (x * y) = y.$$

(IV) holds. Repeatedly using (7),

$$(x * y) * (z * u) = u * (z * (x * y)) = u * (y * (x * z)) = (x * z) * (y * u).$$

(3) holds. Summarizing the above results, $< X; *, 0 >$ is a medial BCI-algebra.

($\Rightarrow$) It is sufficient to verify (5). Since

$$\begin{aligned}(x * 0) * (y * z) &= x * (y * z) \qquad [\text{by } (1)]\\ &= (0 * (0 * x)) * (y * z) \qquad [\text{by } (4)]\\ &= (0 * y) * ((0 * x) * z) \qquad [\text{by } (3)]\\ &= (0 * y) * ((0 * z) * x) \qquad [\text{by } (2)]\\ &= (0 * (0 * z)) * (y * x) \qquad [\text{by } (3)]\\ &= z * (y * x), \qquad [\text{by } (4)]\end{aligned}$$

(5) holds. This completes the proof.

COROLLARY 3. *The class of all of medial BCI-algebras forms a variety, written V(MI).*

Looking at the proof of Theorem 2 it is not difficult to find the following.

COROLLARY 4. *An algebra* $< X; *, 0 >$ *of type (2, 0) is a medial BCI-algebra if and only if it satisfies:*
(7) $x * (y * z) = z * (y * x)$,
(1) $x * 0 = x$,
(III) $x * x = 0$.

PROPOSITION 5. *[1] A variety V is congruence-permutable if and only if there is a term* $p(x, y, z)$ *such that*

$$V \models p(x, x, y) \approx y,$$
$$V \models p(x, y, y) \approx x.$$

COROLLARY 6. *The variety* $V(MI)$ *is congruence-permutable.*

Proof. Let $p(x, y, z) = x * (y * z)$. Then by (4) and (6) we have $p(x, x, y) = y$ and $p(x, y, y) = x$, and so the variety $V(MI)$ is congruence-permutable.

COROLLARY 7. *A BCI-algebra is medial if and only if it is p-semisimple.*

Proof. It is a consequence of [5, Theorem 5].

In [4], W. A. Dudek give the following theorem.

THEOREM 8. *The class of all BCI-algebras with (4) is equationally definable by (4) and (2).*

But this result is incorrect (see [6]). For example, the algebra $< X; *, 0 >$ defined by the following table satisfies conditions (4) and (2) while it is not a BCI-algebra.

*	0	1
0	1	0
1	0	1

It is natural to ask whether the class of all BCI-algebras with (4) can be defined by two identities. Our answer is positive.

THEOREM 9. *For a BCI-algebra* $< X; *, 0 >$, *conditions (4) and (3) are equivalent.*

Proof. Suppose a BCI-algebra $< X; *, 0 >$ satisfies the condition (4). By [5, Theorem 5(27)], we have

$$\begin{aligned}(x * y) * (z * u) &= u * (z * (x * y)) \\ &= u * (y * (x * z)) \\ &= (x * z) * (y * u).\end{aligned}$$

Hence $< X; *, 0 >$ also satisfies (3).

Conversely let $< X; *, 0 >$ satisfies (3); then it is p-semisimple by Corollary 7. Thus it satisfies (4) by [5, Theorem 5(25)].

As an immediate consequence of Theorems 2 and 9 we have the following.

THEOREM 10. *The class of all BCI-algebras with (4) is equationally definable by (5) and (6).*

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