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ON CERTAIN CLASSES OF MULTIVALENT FUNCTIONS

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1. Introduction

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in N = \{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : \; |z| < 1\}$. Let f and g belong to A_p. We denote by $f * g$ the Hadamard product or convolution of $f, g \in A_p$, that is, if

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad \text{and} \quad g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}, \tag{1.2}$$

then

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p}. \tag{1.3}$$

Let

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$$
(1.4)\qquad \begin{aligned} D^{n+p-1}f(z) &= \frac{z^p}{(1-z)^{n+p}} * f(z) \\ &= \frac{z^p\left(z^{n-1}f(z)\right)^{(n+p-1)}}{(n+p-1)!}, \end{aligned}
$$

where n is any integer greater than $-p$.

Goel and Sohi [2] introduced the classes $K_{n,p}$ (*i.e.*, $K_{n,p}$) of functions $f \in A_p$ which satisfy the condition

$$
(1.5)\qquad Re\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right\} > \frac{1}{2} \quad (z \in U).
$$

They proved that $K_{n+1,p} \subset K_{n,p}$ for any integer n greater than $-p$.

Let $R_{n,p}(\alpha)$ denote the classes of functions $f \in A_p$ which satisfy the condition

$$
(1.6)\qquad Re\left\{\frac{z\left(D^{n+p-1}f(z)\right)'}{pD^{n+p-1}f(z)}\right\} > \alpha \quad (z \in U)
$$

for some α $(0 \le \alpha < 1)$. We have $R_{-p+1,p}(\alpha) = S_p^*(\alpha)$, where $S_p^*(\alpha)$ is the well known class of p-valent starlike functions of order α. For $p = 1$, the classes $R_{n,1}(0)$ and $R_{n,1}(\alpha)$ were considered by Singh and Singh [7] and Ahuja [1], respectively.

In this paper, we prove that $R_{n+1,p}(\alpha) \subset R_{n,p}(\alpha)$. Since $R_{-p+1,p}(\alpha)$ is a class of p-valent starlike functions [9], it follows that all functions in $R_{n,p}(\alpha)$ are p-valent. We also investigate some properties of the classes $R_{n,p}(\alpha)$. Furthermore, we obtain some special elements of $R_{n,p}(\alpha)$ by Hadamard product.

2. Some properties of the classes $R_{n+p-1}(\alpha)$

We need the following lemma due to Jack [3] for the proofs of the comming results.

LEMMA 1. *Let w be a nonconstant and analytic function in $|z| < r < 1$, $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r$ at z_0, then $z_0 w'(z_0) = kw(z_0)$, where k is a real number and $k \geq 1$.*

THEOREM 1. *$R_{n+1,p}(\alpha) \subset R_{n,p}(\alpha)$ for any integer n greater than $-p$.*

proof. Let $f \in R_{n+1,p}(\alpha)$. Then

$$Re\left\{\frac{z\,(D^{n+p}f(z))'}{pD^{n+p}f(z)}\right\} > \alpha. \tag{2.1}$$

We have to show that (2.1) implies the inequality

$$Re\left\{\frac{z\,\left(D^{n+p-1}f(z)\right)'}{pD^{n+p-1}f(z)}\right\} > \alpha. \tag{2.2}$$

Define $w(z)$ in U by

$$\frac{z\,\left(D^{n+p-1}f(z)\right)'}{pD^{n+p-1}f(z)} = \frac{1+(2\alpha-1)w(z)}{1+w(z)}. \tag{2.3}$$

Clearly $w(z)$ is analytic, $w(0) = 0$ and $w(z) \neq -1$. Using the identity

$$z\left(D^{n+p-1}f(z)\right)' = (n+p)D^{n+p}f(z) - nD^{n+p-1}f(z), \tag{2.4}$$

the equation (2.3) may be written as

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{(n+p)+(n+p(2\alpha-1))\,w(z)}{(n+p)\,(1+w(z))}. \tag{2.5}$$

Differentiating (2.5) logarithmically, we obtain

$$\frac{z\,(D^{n+p}f(z))'}{pD^{n+p}f(z)} = \frac{1+(2\alpha-1)w(z)}{1+w(z)} - \frac{2(1-\alpha)zw'(z)}{(1+w(z))\,((n+p)+(n+p(2\alpha-1))\,w(z))}. \tag{2.6}$$

We claim that $|w(z)| < 1$. For otherwise, by Lemma 1, there exists $z_0 \in U$ such that

$$z_0 w'(z_0) = kw(z_0), \tag{2.7}$$

where $|w(z_0)| = 1$ and $k \geq 1$. The equation (2.6) in conjugation with (2.7) yields

(2.8)

$$\frac{z_0\,(D^{n+p}f(z_0))'}{pD^{n+p}f(z_0)} = \frac{1+(2\alpha-1)w(z_0)}{1+w(z_0)} - \frac{2(1-\alpha)zw'(z_0)}{(1+w(z_0))\,((n+p)+(n+p(2\alpha-1))\,w(z_0))}.$$

Thus

$$Re\left\{\frac{z_0\,(D^{n+p}f(z_0))'}{pD^{n+p}f(z_0)}\right\} \leq \alpha - \frac{k(1-\alpha)}{2(n+p\alpha)} \leq \alpha, \tag{2.9}$$

which contradicts (2.1) and from (2.3) it follows that $f \in R_{n,p}(\alpha)$.

THEOREM 2. *Let $f \in R_{n,p}(\alpha)$. Then*

$$Re\left\{\frac{D^{n+p-1}f(z)}{z^p}\right\}^{\beta} > \frac{1}{2\beta p(1-\alpha)+1} \quad (z \in U), \tag{2.10}$$

where $0 < \beta \leq \frac{1}{2p(1-\alpha)}$.

Proof. Let $f \in R_{n,p}(\alpha)$, let $\gamma = \frac{1}{2\beta p(1-\alpha)+1}$ and let $w(z)$ be analytic function such that

$$\left\{\frac{D^{n+p-1}f(z)}{z^p}\right\}^{\beta} = \frac{1+(2\gamma-1)w(z)}{1+w(z)}. \tag{2.11}$$

Then $w(0) = 0$ and $w(z) \neq -1$. The theorem will follow if we can show that $|w(z)| < 1$ in U. Now differentiating (2.11) logarithmically, we get

$$\frac{z\left(D^{n+p-1}f(z)\right)'}{pD^{n+p-1}f(z)} = 1 - \frac{2(1-\gamma)zw'(z)}{\beta p(1+w(z))(1+(2\gamma-1)w(z))}. \tag{2.12}$$

We now claim that $|w(z)| < 1$ for $z \in U$. For otherwise, by lemma 1, there exists a point $z_0 \in U$ such that $z_0 w'(z_0) = kw(z_0)$ with $|w(z_0)| = 1$ and $k \geq 1$. Applying this result to (2.12), we obtain

$$Re\left\{\frac{z_0\left(D^{n+p-1}f(z_0)\right)'}{pD^{n+p-1}f(z_0)}\right\} \leq 1 - \frac{k(1-\gamma)}{2\beta p\gamma} \leq \alpha. \tag{2.13}$$

This contradicts the hypothesis that $f \in R_{n,p}(\alpha)$. Hence we conclude that $|w(z)| < 1$ for $z \in U$. This completes the proof of theorem.

Taking $p = 1$, $n = 0$ and $\beta = \frac{1}{2(1-\alpha)}$ in Theorem 2, we obtain the following corollary which was proved by Jack [3].

COROLLARY 1. *Let $f \in S_1^*(\alpha)$. Then*

$$Re\left\{\frac{f(z)}{z}\right\}^{\frac{1}{2(1-\alpha)}} > \frac{1}{2} \quad (z \in U). \tag{2.14}$$

Putting $p = 1$, $n = 0$ and $\beta \neq 1$ in Theorem 2, we have

COROLLARY 2. *Let $f \in S_1^*(\alpha)$ $(\frac{1}{2} \leq \alpha < 1)$. Then*

$$Re\left\{\frac{f(z)}{z}\right\} > \frac{1}{3-2\alpha} \quad (z \in U). \tag{2.15}$$

REMARK. Under the condition of Corollary 2, taking $\alpha = \frac{1}{2}$, we have a result of MacGregor [6].

Taking $p = 1$, $n = 1$ and $\beta = \frac{1}{2}$ in Theorem 2, we obtain the following known result of Strohacker [8].

COROLLARY 3. *Let $f \in A_1$ be such that $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$. Then*

$$Re\left\{\sqrt{f'(z)}\right\} > \frac{1}{2} \quad (z \in U). \tag{2.16}$$

3. Special elements of the classes $R_{n,p}(\alpha)$

In this section, we form special elements of the classes $R_{n,p}(\alpha)$ by the Hadamard product of elements of $R_{n,p}(\alpha)$ and $h_c(z)$, where

$$h_c(z) = \sum_{j=p}^{\infty} \frac{c+p}{c+j} z^j \quad (Re\ c > -p).$$

THEOREM 3. *Let $f \in R_{n,p}(\alpha)$ and $c + p\alpha > 0$. Then*

$$F(z) = (f * h_c)(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt \tag{3.1}$$

belongs to $R_{n,p}(\alpha)$.

Proof. Let $f \in R_{n+p-1}(\alpha)$. From (3.1), we obtain

$$z\left(D^{n+p-1}F(z)\right)' = (p+c)D^{n+p-1}f(z) - cD^{n+p-1}F(z). \tag{3.2}$$

Define $w(z)$, analytic in U by

$$\frac{z\left(D^{n+p-1}F(z)\right)'}{pD^{n+p-1}F(z)} = \frac{1 + (2\alpha - 1)w(z)}{1 + w(z)}. \tag{3.3}$$

Obviously $w(0) = 0$ and $w(z) \neq -1$ for $z \in U$. It is sufficient to show that $|w(z)| < 1$ for $z \in U$. Using the identity (3.2) and taking the logarithmic derivative of (3.3), we get

$$\frac{z\left(D^{n+p-1}f(z)\right)'}{pD^{n+p-1}f(z)} = \frac{1+(2\alpha-1)w(z)}{1+w(z)} - \frac{2(1-\alpha)zw'(z)}{(1+w(z))\left((c+p)+(c+p(2\alpha-1))\,w(z)\right)}. \tag{3.4}$$

The remaining part of the proof is similar to that of Theorem 1.

In case $c = n$, Theorem 3 can be improved as follows.

THEOREM 4. *Let $f \in R_{n,p}(\alpha)$ and let n be any integer greater than $-p$. Then*

$$F(z) = \frac{n+p}{z^n}\int_0^z t^{n-1}f(t)dt \tag{3.5}$$

belongs to $R_{n+1,p}(\alpha)$.

Proof. Let $f \in R_{n,p}(\alpha)$. Applying (2.4) and (3.2), we have

$$D^{n+p-1}f(z) = D^{n+p}F(z). \tag{3.6}$$

Therefore

$$Re\left\{\frac{z\left(D^{n+p}F(z)\right)'}{pD^{n+p}F(z)}\right\} = Re\left\{\frac{z\left(D^{n+p-1}f(z)\right)'}{pD^{n+p-1}f(z)}\right\} > \alpha. \tag{3.7}$$

Hence $F \in R_{n+1,p}(\alpha)$.

THEOREM 5. *Let $f \in A_p$ satisfy the condition*

$$(3.8)\qquad Re\left\{\frac{z\left(D^{n+p-1}f(z)\right)'}{pD^{n+p-1}f(z)}\right\} > \alpha - \frac{1-\alpha}{2(c+p\alpha)} \quad (z \in U),$$

where n is any integer greater than $-p$ and $c + p\alpha > 0$ $(0 \leq \alpha < 1)$. Then $F(z)$ as given by (3.1) belongs to $R_{n,p}(\alpha)$.

The proof of this theorem is similar to that of Theorem 3 and so we omit it.

The following special cases of Theorem 5 represent some improvement on theorems due to Libera [4] in the sense that much weaker assumptions produce the same results.

Taking $p = 1, n = 0$ and $\alpha = 0$ in Theorem 5, we get

COROLLARY 4. *Let $f \in A_1$ be such that $Re\left\{\frac{zf'(z)}{f(z)}\right\} > -\frac{1}{2c}$ $(c > 0)$. Then $F(z)$ is starlike in U, where*

$$(3.9)\qquad F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt.$$

Putting $p = 1, n = 0$ and $\alpha = 0$, Theorem 5 reduces to

COROLLARY 5. *Let $f \in A_1$ be such that $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2c}$ $(c > 0)$. Then $F(z)$ as given by (3.9) above is convex in U.*

We now prove the converse of Theorem 3.

THEOREM 6. *Let $F \in R_{n,p}(\alpha)$ and $c + p\alpha > 0$. Let $f(z)$ be defined as*

$$(3.10)\qquad F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt.$$

Then $f \in R_{n,p}(\alpha)$ in $|z| < R_c = \frac{-((1-\alpha)p+1)+\sqrt{((1-\alpha)p+1)^2+(c+p)(c+2\alpha p-p)}}{c+2\alpha p-p}$.

Proof. Since $F \in R_{n,p}(\alpha)$, we can write

$$(3.11) \qquad \frac{z\left(D^{n+p-1}F(z)\right)'}{pD^{n+p-1}F(z)} = (\alpha + (1-\alpha)u(z)),$$

where $u \in P$, the class of functions with positive real part in U and normalized by $u(0) = 1$. Using the identity (3.2) and differentiating (3.11) logarithmically, we get

(3.12)
$$\left(\frac{z\left(D^{n+p-1}f(z)\right)'}{pD^{n+p-1}f(z)} - \alpha\right)(1-\alpha)^{-1} = u(z) + \frac{zu'(z)}{c + p(\alpha + (1-\alpha)u(z))}.$$

Using the well known estimate $|zu'(z)| \leq \frac{2r}{1-r^2} Re\,\{u(z)\}$ and $Re\,\{u(z)\} \geq \frac{1-r}{1+r}$ $(|z| = r)$, the equation (3.12) yields

$$(3.13) \quad Re\left\{\left(\frac{z\left(D^{n+p-1}f(z)\right)'}{pD^{n+p-1}f(z)} - \alpha\right)(1-\alpha)^{-1}\right\}$$
$$\geq Re\{u(z)\}\left(1 - \frac{2r}{(1-r)\left((c+p\alpha)(1+r) + (1-\alpha)p(1-r)\right)}\right).$$

Now the right hand side of (3.13) is positive provided $r < R_c$. Hence $f \in R_{n,p}(\alpha)$ for $|z| < R_c$.

Taking $p = 1$ and $n = 0$ in Theorem 6, we get the following result.

COROLLARY 6. *Let $F \in S_1^*(\alpha)$ and $c + \alpha > 0$. Let $f(z)$ be defined as (3.9). Then $f \in S_1^*(\alpha)$ for $|z| < \frac{\alpha - 2 + \sqrt{(2-\alpha)^2 + (c+1)(c+2\alpha-1)}}{c+2\alpha-1}$.*

REMARK. Above Corollary 6 is an extension of the result obtained earlier by Libera and Livingston [5] for $c = 1$.

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