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SYSTEMS OF SIMULTANEOUS EQUATIONS OF VECTOR FORMS ON OPERATOR ALGEBRAS

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Let $\mathcal{H}$ be a separable, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For a linear manifold $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$, a *form* on $\mathcal{A}$ is a linear functional on $\mathcal{A}$. For $x,\ y \in \mathcal{H},\ x \otimes y$ denotes the form on $\mathcal{L}(\mathcal{H})$ defined by $x \otimes y(S) = (Sx, y)$ for any $S \in \mathcal{L}(\mathcal{H})$ (cf. [2]). An *elementary form* on a linear manifold $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$ is the restriction $x \otimes y \mid \mathcal{A}$ for $x, y \in \mathcal{H}$. It is well-known that there are several Hausdroff locally convex topologies on $\mathcal{L}(\mathcal{H})$. Recently several functional analysists have been studied systems of simultaneous equations of weak* continuous elementary forms on a singly generated operator algebra (cf. [3]). This study has been applied to invariant subspaces, dilation theory, and reflexivity for contraction operators. In particular, Jung-Kim (cf. [5]) introduced property $(\tau_{m,n})$ which are concerned with the system of simultaneous equations of vector forms and obtained some new dilations of operator algebras related with property $(\tau_{m,n})$. This paper is a sequel study of those in [5].

Throughout this paper the topology τ is one of the following topologies; weak operator topology, operator-normed topology, strong operator topology, weak* topology (or equivalently, ultra-weak operator topology), or ultra-strong operator topology on $\mathcal{L}(\mathcal{H})$. $\mathbf{N}$ denotes the set of natural numbers and $\mathbf{C}$ the complex plane. $\mathcal{A}$ denotes a unital subalgebra of $\mathcal{L}(\mathcal{H})$ (note that the closedness of $\mathcal{A}$ is not considered).

DEFINITION 1. Suppose that m and n are any cardinal numbers such that $1 \le m,\ n \le \aleph_0$ and r is a fixed real number satisfying $r \ge 1$. A subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ has property $(\tau_{m,n}(r))$ if for any τ-continuous

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form $\{\phi_{ij}\}_{\substack{0\le i<m\\ 0\le j<n}}$ on $\mathcal{A}$ and $r < s$, there exist $\{x_i\}_{0\le i<m}$ and $\{y_j\}_{0\le j<n}$ in $\mathcal{H}$ such that $\phi_{ij} = x_i \otimes y_j$ on $\mathcal{A}$,

$$\|x_i\| \le \left(s \sum_{0\le j<n} \|\phi_{ij}\|\right)^{\frac{1}{2}} \quad \text{for} \quad 0\le i<m ,$$

and

$$\|y_j\| \le \left(s \sum_{0\le i<m} \|\phi_{ij}\|\right)^{\frac{1}{2}} \quad \text{for} \quad 0\le j<n .$$

PROPOSITION 2. *Assume that the adjoint operation $\Phi(A) = A^*$ from $\mathcal{A}$ onto $\mathcal{A}^*(=\{A^*|A\in\mathcal{A}\})$ is τ-continuous under the given topology τ in $\mathcal{L}(\mathcal{H})$. Suppose m and n are any cardinal numbers such that $1\le m,n\le\aleph_0$. Then $\mathcal{A}$ has property $(\tau_{m,n}(r))$ if and only if $\mathcal{A}^*$ has property $(\tau_{n,m}(r))$.*

Proof. Let $\{\phi_{ji}\}_{\substack{0\le i<m\\ 0\le j<n}}$ be a system of τ-continuous forms on $\mathcal{A}^*$. Put $\psi_{ij} = \overline{\phi}_{ji}\circ\Phi$ for $0\le i<m, 0\le j<n$, where $\overline{\phi_{ji}}(S) = \overline{\phi_{ji}(S)}$ for S $\in \mathcal{A}^*$. Then ψ_{ij} is τ-continuous form on $\mathcal{A}$. By definition, there exist $\{x_i\}_{0\le i<m}$ and $\{y_j\}_{0\le j<n}$ in $\mathcal{H}$ such that $\psi_{ij} = x_i\otimes y_j$,

$$\|x_i\| \le \left(s \sum_{0\le j<n} \|\psi_{ij}\|\right)^{\frac{1}{2}} \quad \text{for} \quad 0\le i<m$$

and

$$\|y_j\| \le \left(s \sum_{0\le i<m} \|\psi_{ij}\|\right)^{\frac{1}{2}} \quad \text{for} \quad 0\le j<n.$$

So $\phi_{ji}(A^*) = \overline{\psi_{ij}(A)} = (A^*y_j, x_i)$ and $\|\psi_{ij}\| = \|\phi_{ji}\|$.
Moreover, we have

$$\|x_i\| \le \left(s \sum_{0\le j<n} \|\phi_{ji}\|\right)^{\frac{1}{2}} \quad \text{for} \quad 0\le i<m$$

and

$$\|y_j\| \leq \left(s \sum_{0 \leq i < m} \|\phi_{ji}\| \right)^{\frac{1}{2}} \quad \text{for} \quad 0 \leq j < n.$$

Hence $\mathcal{A}^*$ has property$(\tau_{n,m}(r))$. Conversely, we can prove the converse implication by a similar method.

PROPOSITION 3. *If $\mathcal{M}$ is a τ-closed subalgebra with property $(\tau_{m,n}(r))$ for some cardinal numbers m and n with $1 \leq m, n \leq \aleph_0$ and $\mathcal{N}$ is a τ-closed subalgebra of $\mathcal{M}$, then $\mathcal{N}$ has property $(\tau_{m,n}(r))$.*

Proof. Let $\{\phi_{ij}\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ be a system of τ-continuous form on $\mathcal{N}$. Since $\mathcal{A}$ is a locally convex space under the given topology τ, by [4, Proposition 14.13], there exists a system $\{\psi_{ij}\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ of τ-continuous forms on $\mathcal{M}$ such that $\psi_{ij}|\mathcal{N} = \phi_{ij}$ and $\|\psi_{ij}\| = \|\phi_{ij}\|$, $0 \leq i < m$, $0 \leq j < n$. Hence there exist x_i, $y_j \in \mathcal{H}$, $0 \leq i < m$, $0 \leq j < n$, such that $\psi_{ij} = x_i \otimes y_j$,

$$\|x_i\| \leq \left(s \sum_{0 \leq j < n} \|\psi_{ij}\| \right)^{\frac{1}{2}} \quad \text{for} \quad 0 \leq i < m$$

and

$$\|y_j\| \leq \left(s \sum_{0 \leq i < m} \|\psi_{ij}\| \right)^{\frac{1}{2}} \quad \text{for} \quad 0 \leq j < n.$$

Moreover, it follows trivially that $\phi_{ij} = x_i \otimes y_j$,

$$\|x_i\| \leq \left(s \sum_{0 \leq j < n} \|\phi_{ij}\| \right)^{\frac{1}{2}} \quad \text{for} \quad 0 \leq i < m$$

and

$$\|y_j\| \leq \left(s \sum_{0 \leq i < m} \|\phi_{ij}\| \right)^{\frac{1}{2}} \quad \text{for} \quad 0 \leq j < n.$$

Hence $\mathcal{N}$ has property $(\tau_{m,n}(r))$ and the proof is complete.

We write

$$\mathcal{A}^{(n)} = \{\underbrace{A \oplus \cdots \oplus A}_{(n)} \mid A \in \mathcal{A}\},$$

which is called an n-th *ampliation* of $\mathcal{A}$.

PROPOSITION 4. *If $\mathcal{A}$ has property $(\tau_{1,1}(r))$, then an ampliation $\mathcal{A}^{(n)}$ has property $(\tau_{1,n}(r))$ for any cardinal number n with $1 \leq n \leq \aleph_0$.*

Proof. Let $\{\phi_i\}_{0 \leq i < n}$ be a system of τ-continuous forms on $\mathcal{A}^{(n)}$. Define $\psi_i(A) = \phi_i(A^{(n)})$ for any $A \in \mathcal{A}$, $0 \leq i < n$. Then ψ_i is a τ-continuous form on $\mathcal{A}$. So there exist $\{x_i\}_{0 \leq i < n}$ and $\{y_i\}_{0 \leq i < n}$ in $\mathcal{H}$ such that $\psi_i = x_i \otimes y_i$,

$$\|x_i\| \leq (s \ \|\psi_i\|)^{\frac{1}{2}} \text{ for } \ 0 \leq i < n$$

and

$$\|y_i\| \leq (s \ \|\psi_i\|)^{\frac{1}{2}} \text{ for } \ 0 \leq i < n.$$

Set

$$\widetilde{x} = \underbrace{(x_0, x_1, \cdots)}_{(n)}$$

and

$$\widetilde{y}_i = (\overbrace{\underbrace{0, \cdots, 0}_{(i)}, y_i, 0, \cdots}^{(n)}) \quad \text{for} \quad 0 \leq i < n.$$

Then it is easy to show that $\phi_i = \tilde{x} \otimes \widetilde{y}_i$, $0 \leq i < n$

$$\|\widetilde{x}\| = \left(\sum_{0 \leq i < n} \|x_i\|^2 \right)^{\frac{1}{2}} \leq \left(s \sum_{0 \leq i < n} \|\phi_i\| \right)^{\frac{1}{2}}$$

and

$$\|\widetilde{y}_i\| = \|y_i\| \leq (s \ \|\phi_i\|)^{\frac{1}{2}} \text{ for } \ 0 \leq i < n.$$

Hence $\mathcal{A}^{(n)}$ has property $(\tau_{1,n}(r))$.

PROPOSITION 5. *If $\mathcal{A}$ has property $(\tau_{1,n}(r))$ for some cardinal number n with $1 \leq n \leq \aleph_0$, then $\mathcal{A}^{(n)}$ has property $(\tau_{n,n}(r))$.*

Proof. Let $\{\phi_{ij}\}_{0\leq i,j<n}$ be a system of τ-continuous forms on $\mathcal{A}^{(n)}$. Define $\psi_{ij}(A) = \phi_{ij}(A^{(n)})$ for $A \in \mathcal{A}$, $0 \leq i,j < n$. Then ψ_{ij} is a τ-continuous form on $\mathcal{A}$. By hypothesis, for fixed i with $0 \leq i < n$, there exist $x_i \in \mathcal{H}$ and $\{y_{ij}\}_{0\leq j<n}$ in $\mathcal{H}$ such that $\psi_{ij} = x_i \otimes y_{ij}$,

$$\|x_i\| \leq \left(s \sum_{0\leq j<n} \|\psi_{ij}\|\right)^{\frac{1}{2}} \quad \text{for} \quad 0 \leq i < n$$

and

$$\|y_{ij}\| \leq (s\,\|\psi_{ij}\|)^{\frac{1}{2}} \text{ for } \quad 0 \leq j < n.$$

Set

$$\widetilde{x}_i = (\overbrace{\underbrace{0,0,\cdots,0}_{(i)},x_i,0,\cdots}^{(n)}) \quad \text{for} \quad 0 \leq i < n$$

and

$$\widetilde{y}_j = (\overbrace{y_{1j},y_{2j},\cdots}^{(n)}) \quad \text{for} \quad 0 \leq j < n.$$

Then it is easy to show that $\phi_{ij} = \widetilde{x}_i \otimes \widetilde{y}_j$, $0 \leq i,j < n$

$$\|\widetilde{x}_i\| = \|x_i\| \leq \left(s \sum_{0\leq j<n} \|\phi_{ij}\|\right)^{\frac{1}{2}} \text{ for } \quad 0 \leq i < n$$

and

$$\|\widetilde{y}_j\| = \left(\sum_{0\leq i<n} \|y_{ij}\|^2\right)^{\frac{1}{2}} \leq \left(s \sum_{0\leq i<n} \|\phi_{ij}\|\right)^{\frac{1}{2}} \text{ for } \quad 0 \leq j < n.$$

Hence $\mathcal{A}^{(n)}$ has property $(\tau_{n,n}(r))$.

PROPOSITION 6. *If $\mathcal{A}$ has property $(\tau_{1,1}(r))$, then $\mathcal{A}^{(n^2)}$ has property $(\tau_{n,n}(r))$ for any cardinal number n with $1 \le n \le \aleph_0$.*

Proof. By Proposition 4 and 5 this proof is simple, since $\left(\mathcal{A}^{(n)}\right)^{(n)}$ is identified with $\mathcal{A}^{(n^2)}$.

We now consider a sufficient condition for property $(\tau_{1,n})$. First, we start from the following definitions.

DEFINITION 7. [1]. If $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, then $Cyc(\mathcal{A}, x)$ denotes the smallest subspace of $\mathcal{H}$ that contains x and invariant for every S in $\mathcal{A}$.

DEFINITION 8. [5]. Suppose m and n are any cardinal numbers with $1 \le m, n \le \aleph_0$. A subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ has property $(\tau_{m,n})$ if for any system $\{\phi_{ij}\}_{\substack{0\le i<m \\ 0\le j<n}}$ on $\mathcal{A}$ of τ-continuous forms, there exist $\{x_i\}_{0\le i<m}$ and $\{y_j\}_{0\le j<n}$ in $\mathcal{H}$ such that $\phi_{ij} = x_i \otimes y_j$ on $\mathcal{A}$.

For a subalgebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$, we write $\widetilde{\mathcal{H}} = \sum \oplus_{i=1}^{n} \mathcal{H}_i$ and $\widetilde{A} = \sum \oplus_{i=1}^{n} A_i$, where $\mathcal{H}_i = \mathcal{H}$ and $A_i = A \in \mathcal{A}$, $1 \le i \le n$. And we denote $\widetilde{\mathcal{A}} = \{\widetilde{A} \mid A \in \mathcal{A}\}$.

THEOREM 9. *If a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ has property $(\tau_{1,1})$ and for each $n \in \mathrm{N}$ and $\widetilde{x} \in \widetilde{\mathcal{H}}$, there exist an element x in $\mathcal{H}$ and a unitary operator*
$U : Cyc(\widetilde{\mathcal{A}}, \widetilde{x}) \longrightarrow Cyc(\mathcal{A}, x)$ such that

$$U^*(A|\, Cyc(\mathcal{A}, x))U \;=\; \widetilde{A} \mid Cyc(\widetilde{\mathcal{A}}, \widetilde{x}),$$

then $\mathcal{A}$ has property $(\tau_{1,n})$.

Proof. By [5, Proposition 2.4 (c)], for τ-continuous form ϕ_i on $\mathcal{A}$, there exist $\widetilde{x}$ and $\widetilde{y}_i$ in $\widetilde{\mathcal{H}}$ such that $\phi_i(A) = (\widetilde{A}\widetilde{x}, \widetilde{y}_i)$, $0 \le i < n$. Let $\mathcal{M} = Cyc(\widetilde{\mathcal{A}}, \widetilde{x})$ and $v_i = UP_{\mathcal{M}}\widetilde{y}_i$ for $0 \le i < n$, where $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$. Since

$$\begin{aligned}(\widetilde{A}\,\widetilde{x}, \widetilde{y}_i) &= (\widetilde{A}\,\widetilde{x}, P_{\mathcal{M}}\widetilde{y}_i) \\ &= (\widetilde{A}\widetilde{x}, U^* v_i) \\ &= (AU\widetilde{x}, v_i)\end{aligned}$$

for any A $\in \mathcal{A}$, $\phi_i(A) = (AU\widetilde{x}, v_i)$ for $0 \le i < n, A \in \mathcal{A}$. Hence $\mathcal{A}$ has property $(\tau_{1,n})$.

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