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AN EXTREME POSITIVE LINEAR OPERATOR ON M_n WHICH MAPS AN EXTREME POINT TO A NON–EXTREME POINT

BYUNG SOO MOON

1. Introduction

We denote M_n for the set of all $n \times n$ complex matrices and E_n for the Hermitian part of M_n. Thus, E_n is the real ordered space of all $n \times n$ Hermitian matrices with the positive cone consisting of all elements having nonnegative eigenvalues. A linear operator T from E_n to E_m is positive if $T(P) \geq 0$ whenever $P \geq 0$, and T is extreme if $S = \lambda T$ for some $\lambda \geq 0$ whenever $0 \leq S \leq T$.

A linear operator which maps every extreme point $\mathbf{xx}^* \in E_n$ to either 0 or $\mathbf{yy}^* \in E_m$ will be called a 'simple' extreme linear operator.

It is proved in [1] and [2] that a positive linear operator on E_2 or from E_2 to E_3 is extreme if and only if it is a simple extreme linear operator. In [3], it is proved that every positive linear operator on E_2 is a sum of simple extreme positive linear operators.

Choi and Lam [4 ; Theorem 4.4] gave an example of a non-square extreme semidefinite biquadratic real polynomials. In the context of positive linear operators on M_n, 'simple' extreme operators correpond to (absolute) squares of bilinear homogeneous polynomials when we consider $\mathbf{z}^*T(\mathbf{xx}^*)\mathbf{z}$ be the equivalent semidefinite form corresponding to a positive linear operator T on M_n.

In this paper, we give an example of a non-square extreme semidefinite biquadratic in complex setting, i.e. a positive semidefinite complex polynomial which is not a sum of absolute squares of homogeneous bilinear forms. In our terms, it will be an example of an extreme positive linear operator which maps an extreme point of the positive cone in M_n to a non-extreme point in M_n.

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We write

$$\mathbf{x}\mathbf{x}^* = \begin{bmatrix} r_1^2 & r_1r_2e^{i\theta_1} & r_1r_3e^{i\theta_2} \\ & r_2^2 & r_2r_3e^{i\theta} \\ & & {r_3}^2 \end{bmatrix}$$

for $\mathbf{x} \in C^3$ where $\theta = \theta_2 - \theta_1$, and let

$$T(\mathbf{x}\mathbf{x}^*) = \begin{bmatrix} r_3^2 I & r_1r_3(\mathbf{e}_1\cos\,\theta_2 + \mathbf{e}_2\sin\,\theta_2) + ir_2r_3(\mathbf{e}_2\cos\,\theta + \mathbf{e}_1\sin\theta \\ & r_1^2 + r_2^2 \end{bmatrix}$$

where I is the identity in E_2, then it is routine to verify that $T \geq 0$. The linear operator defined above will always be denoted by T and all the linear operators we consider in this paper will be assumed to be from E_3 to E_3.

Note that there is a natural extension of every positive linear operator on E_n to a positive linear operator on M_n.

Since $T(\mathbf{e}_3\mathbf{e}_3^T) = I_2$ where I_2 is the identity in E_2, T maps an extreme point to a non-extreme point. We have to prove that T is extreme. But, first we prove that T is not the sum of simple extreme positive linear operators, which gives us some assurance that T may be extreme.

2. T is not a Sum of Simple Extreme Operators

In the following, we will use E_{ii} for $\mathbf{e}_i\mathbf{e}_i^T, E_{kl}$ for $(\mathbf{e}_k\mathbf{e}_l^T + \mathbf{e}_l\mathbf{e}_k^T)$, and $\tilde{E}_{kl}$ for $i\mathbf{e}_k\mathbf{e}_l^T - i\mathbf{e}_l\mathbf{e}_k^T, k \neq l$.

LEMMA 2.1. *If S is a linear operator with $0 \leq S \leq T$, then $S(E_{13})$, $S(\tilde{E}_{13}), S(E_{23}), S(\tilde{E}_{23})$ are all of the form* $\begin{bmatrix} 0 & \mathbf{a} \\ & 0 \end{bmatrix}$ *for some* $\mathbf{a} \in \mathbf{C}^2$.

Proof. Note that we must have $S(E_{11}) = tE_{33}, S(E_{33}) = \begin{bmatrix} P & 0 \\ & 0 \end{bmatrix}$ for some $t \geq 0$, $0 \leq P \in E_2$. Let

$$S(E_{13}) = \begin{bmatrix} A & \mathbf{a} \\ & \lambda \end{bmatrix},\ S(\tilde{E}_{13}) = \begin{bmatrix} B & \mathbf{b} \\ & \mu \end{bmatrix},$$

then

$$S\begin{bmatrix} 1 & 0 & re^{i\theta} \\ & 0 & 0 \\ & & r^2 \end{bmatrix} = \begin{bmatrix} r^2P + r(A\cos\theta + B\sin\theta) & r(\mathbf{a}\cos\theta + \mathbf{b}\sin\theta) \\ & t + r(\lambda\cos\theta + \mu\sin\,\theta) \end{bmatrix} \geq 0.$$

Hence $t + r(\lambda\cos\theta + \mu\sin\theta) \geq 0$ for all $r \geq 0, \theta \in \mathbf{R}$, from which we obtain $\lambda = \mu = 0$. Also, from $rP + A\cos\theta + B\sin\theta \geq 0$ for all $r \geq 0$ and $\theta \in \mathbf{R}$, we obtain $A = B = 0$. A similar proof for $S(E_{23})$ and $S(\tilde{E}_{23})$ is omitted.

LEMMA 2.2. *If S is a simple extreme positive linear operator with $0 \leq S \leq T$, then*

$$S(\mathbf{xx}^*) = \begin{bmatrix} r_3^2\mathbf{qq}^* & r_3(\lambda r_1 e^{i\theta_2} + \mu r_2 e^{i\theta})\mathbf{q} \\ & |\lambda|^2 r_1^2 + |\mu|^2 r_2^2 + r_1 r_2(f\cos\theta_1 + g\sin\theta_1) \end{bmatrix}$$

where $\mathbf{xx}^* = \begin{bmatrix} r_1^2 & r_1 r_2 e^{i\theta_1} & r_1 r_3 e^{i\theta_2} \\ & r_2^2 & r_2 r_3 e^{i\theta} \\ & & r_3^2 \end{bmatrix}$, $\theta = \theta_2 - \theta_1$, $f = \lambda\bar{\mu} + \bar{\lambda}\mu$, *and* $g = i(\lambda\bar{\mu} - \bar{\lambda}\mu)$ *or*

$$S(\mathbf{xx}^*) = \begin{bmatrix} r_3^2\mathbf{qq}^* & r_3(\lambda r_1 e^{-i\theta_2} + \mu r_2 e^{-i\theta})\mathbf{q} \\ & |\lambda|^2 r_1^2 + |\mu|^2 r_2^2 + r_1 r_2(f'\cos\theta_1 + g'\sin\theta_1) \end{bmatrix}$$

where $f' = f, g' = -g$.

Proof. Note the $S(E_{33}) = \begin{bmatrix} \mathbf{qq}^* & 0 \\ & 0 \end{bmatrix}$ for some $\mathbf{q} \in C^2$ sine $S(E_{33})$ is extreme by assumption. By Lemma 2.1, $S(\mathbf{xx}^*)$ is of the form

$$S(\mathbf{xx}^*) = \begin{bmatrix} r_3^2\mathbf{qq}^* & r_1 r_3(\mathbf{a}\cos\theta_2 + \mathbf{b}\sin\theta_2) + r_2 r_3(\mathbf{c}\cos\theta + \mathbf{d}\sin\theta) \\ & fr_1^2 + gr_2^2 + r_1 r_2(\gamma\cos\theta_1 + \delta\sin\theta_1) \end{bmatrix}.$$

Since $S(\mathbf{xx}^*)$ is either 0 or extreme in E_3 for all $\mathbf{x} \in \mathbf{C}^3$, we have

$$\begin{aligned} &(fr_1^2 + gr_2^2 + r_1 r_2(\gamma\cos\theta_1 + \delta\sin\theta_1))\mathbf{qq}^* \\ &= \{r_1(\mathbf{a}\cos\theta_2 + \mathbf{b}\sin\theta_2) + r_2(\mathbf{c}\cos\theta + \mathbf{d}\sin\theta)\} \\ &\quad \cdot \{r_1(\mathbf{a}\cos\theta_2 + \mathbf{b}\sin\theta_2) + r_2(\mathbf{c}\cos\theta + \mathbf{d}\sin\theta)\}^*. \end{aligned} \tag{1}$$

By comparing the coefficients of r_1^2, we obtain $f\mathbf{qq}^* = \cos^2\theta_2\mathbf{aa}^* + \sin^2\theta_2\mathbf{bb}^* + \sin\theta_2\cos\theta_2(\mathbf{ab}^* + \mathbf{ba}^*)$ for all $\theta_2 \in \mathbf{R}$. Thus, we have

$$\mathbf{aa}^* = \mathbf{bb}^* = f\mathbf{qq}^*, \ \mathbf{ab}^* + \mathbf{ba}^* = 0. \tag{2}$$

Similarly, from the cofficients of r_2^2, we obtain

$$\text{(3)} \qquad \mathbf{cc}^* = \mathbf{dd}^* = g\mathbf{qq}^*, \ \mathbf{cd}^* + \mathbf{dc}^* = 0$$

From (2) and (3), we have $\mathbf{a} = \lambda\mathbf{q}, \mathbf{b} = \alpha\mathbf{q}, \mathbf{c} = \mu\mathbf{q}, \mathbf{d} = \beta\mathbf{q}$ with $|\lambda|^2 = |\alpha|^2 = f, |\mu|^2 = |\beta|^2 = g$. Let $\alpha = \lambda e^{i\sigma}, \beta = \mu e^{i\tau}$ and substitute these into (1) with $r_2 = 0$ to obtain $1 = |\cos\theta_2 + e^{i\sigma}\sin\theta_2|^2$ for all $\theta_2 \in \mathbf{R}$, i.e. $1 = 1 + 2\cos\sigma\sin\theta_2\cos\theta_2$. Hence, we have $\cos\sigma = 0$, i.e. $e^{i\sigma} = \pm i$. Similarly, with $r_1 = 0$ in (1), we obtain $e^{i\tau} = \pm i$.

Consider the case with $e^{i\sigma} = i, e^{i\tau} = -i$, then we have

$$S(\mathbf{xx}^*) = \begin{bmatrix} r_3^2\mathbf{qq}^* & r_3(\lambda r_1 e^{i\theta_2} + \mu r_2 e^{-i\theta}) \\ & |\lambda|^2 r_1^2 + |\mu|^2 r_2^2 + r_1 r_2(\cos\theta_1 + \delta\sin\theta_1) \end{bmatrix} \geq 0$$

Hence, (1) becomes $|\lambda|^2 r_1^2 + |\mu|^2 r_2^2 + r_1 r_2(\gamma\cos\theta_1 + \delta\sin\theta_1) = |\lambda r_1 e^{i\theta_2} + \mu r_2 e^{-i\theta}|^2 = |\lambda|^2 r_1^2 + |\mu|^2 r_2^2 + r_1 r_2(\bar{\lambda}\mu e^{-i(\theta+\theta_2)} + \lambda\bar{\mu}e^{i(\theta+\theta_2)})$ for all $\theta_1, \theta_2 \in \mathbf{R}$, with $\theta = \theta_2 - \theta_1$. Therefore, we must have $\lambda = \mu = \gamma = \delta = 0$, i.e. $S = 0$. Similarly, we obtain $S = 0$ when $e^{i\sigma} = -i, e^{i\tau} = i$. Thus, for a nontrival S, we must have $e^{i\sigma} = e^{i\tau}$ which is either i or $-i$ and the result follows.

Theorem 2.3. *T is not the sum of simple extreme operators.*

Proof. Suppose T is a sum of simple extreme poitive linear operators, then by Lemma 2.2, we must have

$$T(\mathbf{xx}^*) = \sum_{i=1}^{m} \begin{bmatrix} r_3^2\mathbf{q}_i\mathbf{q}_i^* & (\lambda_i r_1 e^{i\theta_2} + \mu_i r_2 e^{i\theta}) r_3\mathbf{q}_i \\ & |\lambda_i|^2 r_1^2 + |\mu_i|^2 r_2^2 + r_1 r_2(\gamma_i\cos\theta_1 + \delta_i\sin\theta_1) \end{bmatrix}$$
$$+ \sum_{j=m+1}^{m+n} \begin{bmatrix} r_3^2\mathbf{q}_j\mathbf{q}_j^* & (\lambda_j r_1 e^{-i\theta_2} + \mu_j r_2 e^{-i\theta}) r_3\mathbf{q}_j \\ & |\lambda_j|^2 r_1^2 + |\mu_j|^2 r_2^2 + r_1 r_2(\gamma_j'\cos\theta_1 + \delta_j'\sin\theta_1) \end{bmatrix}.$$

First, we consider the case with $m \geq 1, n \geq 1$. By comparing the correponding elements, we obtain

$$\text{(4)}\quad \sum_{i=1}^{m} \lambda_i \mathbf{q}_i + \sum_{j=m+1}^{m+n} \lambda_j \mathbf{q}_j = \mathbf{e}_1, \sum_{i=1}^{m} \lambda_i \mathbf{q}_i - \sum_{j=m+1}^{m+n} \lambda_j \mathbf{q}_j = -i\mathbf{e}_2$$

$$\text{(5)}\quad \sum_{i=1}^{m} \mu_i \mathbf{q}_i + \sum_{j=m+n}^{m+1} \mu_j \mathbf{q}_j = i\mathbf{e}_2, \sum_{i=1}^{m} \mu_i \mathbf{q}_i - \sum_{j=m+1}^{m+n} \mu_j \mathbf{q}_j = \mathbf{e}_1$$

$$\text{(6)}\quad \sum_{i=1}^{m} \mathbf{q}_i \mathbf{q}_i^* + \sum_{j=m+1}^{m+n} \mathbf{q}_j \mathbf{q}_j^* = I_2$$

$$\text{(7)}\quad \sum_{i=1}^{m+n} |\lambda_i|^2 = 1, \sum_{i=1}^{m+n} |\mu_i|^2 = 1$$

From (4), we obtain $\sum_{i=1}^{m} \lambda_i \mathbf{q}_i = \frac{1}{2}\binom{1}{-i} \equiv \xi_1$, $\sum_{j=m+1}^{m+n} \lambda_j \mathbf{q}_j = \frac{1}{2}\binom{1}{i} \equiv \xi_2$. Note that $\{\xi_1, \xi_2\}$ is linearly independent and hence we may write $\mathbf{q}_j = a_{j1}\xi_1 + a_{j2}\xi_2, j = 1, 2, \cdots, m+n$. Then the above relations become

$$\left(\sum_{i=1}^{m} \lambda_i a_{i1}\right) \xi_1 + \left(\sum_{i=1}^{m} \lambda_i a_{i2}\right) \xi_2 = \xi_1,$$

$$\left(\sum_{i=m+1}^{m+n} \lambda_j a_{j1}\right) \xi_1 + \left(\sum_{j=m+1}^{m+n} \lambda_j a_{j2}\right) \xi_2 = \xi_2$$

from which we obtain

$$\text{(8)}\quad \sum_{i=1}^{m} \lambda_i a_{i1} = \sum_{j=m+1}^{m+n} \lambda_j a_{j2} = 1, \sum_{i=1}^{m} \lambda_i a_{i2} = \sum_{j=m+1}^{m+n} \lambda_j a_{j1} = 0$$

Similary from (5), we obtain

$$\text{(9)}\quad \sum_{i=1}^{m} \mu_i a_{i1} = \sum_{j=m+1}^{m+n} \mu_j a_{j2} = 0, \sum_{i=1}^{m} \mu_i a_{i2} = -\sum_{j=m+1}^{m+n} \mu_j a_{j1} = 1$$

Note that $q_{j1} = \frac{1}{2}(a_{j1} + a_{j2}), q_{j2} = \frac{i}{2}(-a_{j1} + a_{j2})$ from $\mathbf{q}_j = a_{j1}\xi_1 + a_{j2}\xi_2$ and hence, we have $|q_{j1}|^2 = \frac{1}{4}(|a_{j1}|^2 + |a_{j2}|^2 + a_{j1}\bar{a}_{j1} + \bar{a}_{j1}a_{j2}), |q_{j2}|^2 =$

$\frac{1}{4}(|a_{j1}|^2+|a_{j2}|^2-a_{j1}\bar{a}_{j2}-\bar{a}_{j1}a_{j2})$. But from (6), we have $\sum_{j=1}^{m+n}|q_{jk}|^2=1$ for $k=1,2$. Therefore, we have

$$(10)\qquad \sum_{j=1}^{m+n}|a_{j1}|^2+\sum_{j=1}^{m+n}|a_{j2}|^2=4,\ \sum_{j=1}^{m+n}(a_{j1}\bar{a}_{j2}+\bar{a}_{j1}a_{j2})=0$$

Now, from (7), $\sum_{i=1}^{m}|\lambda_i|^2\le 1$ and hence from (8), we must have $\sum_{i=1}^{m}|a_{i1}|^2\ge 1, \sum_{j=m+1}^{m+n}|a_{j2}|^2\ge 1$. Similarly, from (7) and (9) we obtain $\sum_{i=1}^{m}|a_{i2}|^2\ge 1, \sum_{j=m+1}^{m+n}|a_{j1}|^2\ge 1$. Applying these results to (10), we obtain

$$\sum_{j=1}^{m}|a_{j1}|^2=\sum_{j=1}^{m}|a_{j2}|^2=\sum_{j=m+1}^{m+n}|a_{j1}|^2=\sum_{j=m+1}^{m+n}|a_{j2}|^2=1.$$

Therefore, $(\bar{\lambda}_i)_{i=1}^{m}$ and $(a_{j1})_{j=1}^{m}$ are m-vectors of norm less than or equal to 1 with inner product of value 1. Thus we must have $(\bar{\lambda}_i)=(a_{j1})$, but this would imply $\lambda_i=0$ for $i\ge m+1$ which is contrary to the second equality of (8).

Next, we consider the case with $n=0$. Then the relations (4) must still hold without the second summation terms, i.e. $\sum_{i=1}^{m}\lambda\mathbf{q}_i=\mathbf{e}_1, \sum_{i=1}^{m}\lambda\mathbf{q}_i=-ie_2$ which is certainly not possible. A similar arguement can be applied so that $m\neq 0$.

3. The Positive Linear Operator T is Extreme

LEMMA 3.1. *Let*

$$S(\mathbf{x}\mathbf{x}^*)=\begin{bmatrix} dr_3^2I & r_1r_3(\xi_1 cos\ \theta_2+\xi_2 sin\ \theta_2)+r_2r_3(\eta_1 cos\ \theta+\eta_2 sin\ \theta) \\ & ar_1^2+br_2^2+r_1r_2(f cos\ \theta_1+g sin\ \theta_1)\end{bmatrix}\ge 0,$$

where $\xi_i,\eta_i\in C^2$. *If rank* $(S(\mathbf{x}\mathbf{x}^*))\le 2$ *for all* $\mathbf{x}\in C^3$ *or if* $0\le S\le T$, *then*

(1) $\xi_1^*\xi_1=\xi_2{}^*\xi_2=ad,\ \xi_1^*\xi_2+\xi_2^*\xi_1=0,$
(2) $\eta_1^*\eta_1=\eta_2^*\eta_2=bd,\ \eta_1^*\eta_2+\eta_2^*\eta_1=0,$
(3) $\xi_1^*\eta_1+\eta_1^*\xi_1=\xi_2^*\eta_2+\eta_2^*\xi_2=fd,$
(4) $\xi_1^*\eta_2+\eta_2^*\xi_1=-(\xi_2^*\eta_1+\eta_1^*\xi_2)=-gd.$

Proof. Note that rank $(T(\mathbf{xx}^*)) \leq 2$ since for each $\mathbf{x} \neq 0, T(\mathbf{xx}^*)$ can be written as a sum of two extreme elements; one in E_2 and the other in E_3. Hence, if $0 \leq S \leq T$, then rank $(S(\mathbf{xx}^*)) \leq 2$ for all $\mathbf{x} \in C^3$. Therefore, by [5; Theorem 4, p47] we have for all $r_1, r_2 \geq 0, \theta_1, \theta_2 \in R$,

$$\begin{aligned} &d(ar_1^2 + br_2^2 + r_1r_2(f\cos\theta_1 + g\sin\,\theta_1)) \\ =&\{r_1(\xi_1^*\cos\,\theta_2 + \xi_2^*\sin\,\theta_2) + r_2(\eta_1^*\cos\,\theta + \eta_2^*\sin\,\theta)\} \\ &\cdot \{r_1(\xi_1\cos\,\theta_2 + \xi_2\sin\,\theta_2) + r_2(\eta_1\cos\,\theta + \eta_2\sin\,\theta)\}. \end{aligned}$$

We take $r_1 = 1, r_2 = 0, \theta_1 = 0$ (i.e. $\theta = \theta_2$) to obtain

$$ad = \xi_1^*\xi_1\cos^2\theta_2 + \xi_2^*\xi_2\sin^2\theta_2 + (\xi_1^*\xi_2 + \xi_2^*\xi_1)\sin\,\theta_2\cos\,\theta_2$$

for all $\theta_2 \in \mathbf{R}$ and hence (1) follows. Similarly, with $r_1 = 0, \theta_1 = 0$ we obtain (2).

Substituting (1) and (2) into the above equation, we get

$$\begin{aligned} d(f\cos\theta_1 + g\sin\,\theta_1) = &(\xi_1^*\eta_1 + \eta_1^*\xi_1)\cos\,\theta_2\cos\,\theta + (\xi_1^*\eta_2 + \eta_2^*\xi_1)\cos\,\theta_2\sin\,\theta \\ &+ (\xi_2^*\eta_1 + \eta_1^*\xi_2)\sin\,\theta_2\cos\,\theta + (\xi_2^*\eta_2 + \eta_2^*\xi_2)\sin\,\theta_2\sin\,\theta. \end{aligned}$$

We take $\theta = 0$ (i.e. $\theta_1 = \theta_2$) and $\theta = \frac{\pi}{2}$ to obtain (3) and (4).

Corollary 3.2. *Let*

$$S(\mathbf{xx}^*) = \begin{bmatrix} r_3^2D & r_1r_3(\xi_1 cos\,\theta_2 + \xi_2 sin\,\theta_2) + r_2r_3(\eta_1 cos\,\theta + \eta_2 sin\,\theta) \\ & ar_1^2 + br_2^2 + r_1r_2(f cos\,\theta_1 + g sin\,\theta_1) \end{bmatrix}$$

where $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$ *with* $d_1 \neq 0, d_2 \neq 0$, *and let* $U = \begin{bmatrix} U_0 & 0 \\ 0 & 1 \end{bmatrix}$ *where* $U_0 = (\mathbf{u}_1, \mathbf{u}_2)$ *is a unitary matrix. If* $0 \leq S \leq U \circ T$, *where* $U \circ T$ *is the composition of* U *and* T, *then the following are satisfied.*

(1) $\langle\xi_1, \xi_1\rangle = \langle\xi_2, \xi_2\rangle = a, Re\langle\xi_1, \xi_2\rangle = 0$
(2) $\langle\eta_1, \eta_1\rangle = \langle\eta_2, \eta_2\rangle = b, Re\langle\eta_1, \eta_2\rangle = 0$
(3) $Re\langle\xi_1, \eta_1\rangle = Re\langle\xi_2, \eta_2\rangle = \frac{1}{2}f$
(4) $Re\langle\xi_2, \eta_1\rangle = -Re\langle\xi_1, \eta_2\rangle = \frac{1}{2}g$

where $\langle \mathbf{z},\mathbf{w}\rangle = \frac{1}{d_1}\bar{z}_1 w_1 + \frac{1}{d_2}\bar{z}_2 w_2$ *whenever* $\mathbf{z} = \binom{z_1}{z_2}$, $\mathbf{w} = \binom{w_1}{w_2}$. *And if* $d_1 \neq 1, d_2 \neq 1$ *then*

(5) $\langle\langle \mathbf{u}_1 - \xi_1, \mathbf{u}_1 - \xi_1\rangle\rangle = \langle\langle \mathbf{u}_2 - \xi_2, \mathbf{u}_2 - \xi_2\rangle\rangle = 1 - a$,
$Re\langle\langle \mathbf{u}_1 - \xi_1, \mathbf{u}_2 - \xi_2\rangle\rangle = 0$

(6) $\langle\langle i\mathbf{u}_2 - \eta_1, i\mathbf{u}_2 - \eta_1\rangle\rangle = \langle\langle i\mathbf{u}_1 - \eta_2, i\mathbf{u}_1 - \eta_2\rangle\rangle = 1 - b$,
$Re\langle\langle i\mathbf{u}_2 - \eta_1, i\mathbf{u}_1 - \eta_2\rangle\rangle = 0$

(7) $Re\langle\langle \mathbf{u}_1 - \xi_1, \mathbf{u}_2 - \xi_2\rangle\rangle = Re\langle\langle i\mathbf{u}_2 - \eta_1, i\mathbf{u}_1 - \eta_2\rangle\rangle = \frac{1}{2}f$

(8) $Re\langle\langle \mathbf{u}_2 - \xi_2, i\mathbf{u}_2 - \eta_1\rangle\rangle = -Re\langle\langle \mathbf{u}_1 - \xi_1, i\mathbf{u}_1 - \eta_2\rangle\rangle = \frac{1}{2}g$,

where $\langle\langle \mathbf{z},\mathbf{w}\rangle\rangle = \frac{1}{1-d_1}\bar{z}_1 w_1 + \frac{1}{1-d_2}\bar{z}_2 w_2$.

Proof. For $\mathbf{z} \in C^3$ with $\mathbf{z}^T = (z_1, z_2, z_3)$ where $z_i \neq 0,\ i = 1,2,3$, we define

$$S_{\mathbf{z}} = \begin{bmatrix} a & \alpha & \beta \\ & b & \gamma \\ & & c \end{bmatrix} = \begin{bmatrix} |z_1|^2 a & z_1\bar{z}_2\alpha & z_2\bar{z}_3\beta \\ & |z_2|^2 b & z_2\bar{z}_3\gamma \\ & & |z_3|^2 c \end{bmatrix},$$

then $S_{\mathbf{z}}$ is a one-to-one strongly positive linear operator, i.e. both $S_{\mathbf{z}}$ and $S_{\mathbf{z}}^{-1}$ are positive. Let $\mathbf{d}^T = \left(\frac{1}{\sqrt{d_1}}, \frac{1}{\sqrt{d_2}}, 1\right)$ and let $S_1 = S_{\mathbf{d}} \circ S, T_1 = S_{\mathbf{d}} \circ U \circ T$. Then we have $0 \leq S_1 \leq T_1$ where

$$S_1(\mathbf{x}\mathbf{x}^*) = \begin{bmatrix} r_3^2 I & r_1r_3(\xi_1'\cos\theta_2 + \xi_2'\sin\theta_2) + r_2r_3(\eta_1'\mathbf{e}_1\cos\theta + \eta_2'\sin\theta) \\ & ar_1^2 + br_2^2 + r_1r_2(f\cos\theta_1 + g\sin\theta_1) \end{bmatrix}$$

with $\xi_j'^T = \left(\frac{\xi_{1j}}{\sqrt{d_1}}, \frac{\xi_{2j}}{\sqrt{d_2}}\right)$, $\eta_j'^T = \left(\frac{\eta_{1j}}{\sqrt{d_1}}, \frac{\eta_{2j}}{\sqrt{d_2}}\right)$. Now, we apply Lemma 3.1 to obtain $\xi_1'^*\xi_1' = \xi_2'^*\xi_2' = a, \xi_1'^*\xi_2' + \xi_2'^*\xi_1' = 0$, etc which can be written as $\langle \xi_1, \xi_1\rangle = \langle \xi_2, \xi_2\rangle = a, Re\langle \xi_1, \xi_2\rangle = 0$ and so forth. Thus, the relations (1) through (4) are obtained.

To prove the relations (5) through (8), we consider $R = U \circ T - S$ and repeat the above process.

LEMMA 3.3. *Let*

$$S(\mathbf{x}\mathbf{x}^*) = \begin{bmatrix} r_3^2 D & r_1r_3(\xi_1\cos\theta_2 + \xi_2\sin\theta_2) + r_2r_3(\eta_1\cos\theta + \eta_2\sin\theta) \\ & ar_1^2 + br_2^2 + r_1r_2(f\cos\theta_1 + g\sin\theta_1) \end{bmatrix}$$

where $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$. *If* $0 \leq S \leq T$ *then* $d_1 = d_2 = a$

Proof. Note that we have $0 \leq \lambda S \leq T$ for all $0 \leq \lambda \leq 1$ and hence if $d_1 \neq 1, d_2 \neq 1$ then by Lamma 3.2, we have

$$\langle\langle \mathbf{e}_1 - \lambda\xi_1, \mathbf{e}_1 - \lambda\xi_1 \rangle\rangle = \langle\langle \mathbf{e}_2 - \lambda\xi_2, \mathbf{e}_2 - \lambda\xi_2 \rangle\rangle = 1 - \lambda a,$$

$$i.e., \quad \frac{|1-\lambda z_1|^2}{1-\lambda d_1} + \frac{\lambda^2 |w_1|^2}{1-\lambda d_2} = 1 - \lambda a$$

where $\xi_1^T = (z_1, w_1), \xi_2^T = (z_2, w_2)$ for all $0 \leq \lambda \leq 1$. Expanding this out, we have $(1-\lambda a)\{1-\lambda(d_1+d_2)+\lambda^2 d_1 d_2\} = (1-\lambda d_2)\{1+\lambda^2|z_1|^2 - \lambda(z_1+\bar{z}_1)\} + (1-\lambda d_1)\{\lambda^2|w_1|^2\}$. By comparing the coefficients of λ, we have $a + d_1 + d_2 = d_2 + z_1 + \bar{z}_1$ and from the cofficients of $\lambda^2, |z_1|^2 + |w_1|^2 + d_2(z_1 + \bar{z}_1) = d_1 d_2 + ad_1 + ad_2$. From these relations, we get $|z_1|^2 + |w_1|^2 = ad$, and hence we have $Re(z_1) = \frac{1}{2}(a + d_1) \leq |z_1| \leq \sqrt{ad_1}$ from which we obtain $a = d_1, z_1 = \bar{z}_1 = |z_1|, w_1 = 0$. Similarly, from $\langle\langle \mathbf{e}_2 - \lambda\xi_2, \mathbf{e}_2 - \lambda\xi_2 \rangle\rangle = 1 - \lambda a$, we obtain $a = d_2$.

Now, assume $d_1 = 1, d_2 \neq 1$. Then we have, for $r_3 = 1$

$$S(\mathbf{xx}^*) = \begin{bmatrix} 1 & 0 & r_1(z_1\cos\theta_2 + z_2\sin\theta_2) + r_2(\alpha_1\cos\theta + \alpha_2\sin\theta) \\ & d_2 & r_1(w_1\cos\theta_2 + w_2\sin\theta_2) + r_2(\beta_1\cos\theta + \beta_2\sin\theta) \\ & & ar_1^2 + br_2^2 + r_1 r_2(f\cos\theta_1 + g\sin\theta_1) \end{bmatrix}$$

$$\leq \begin{bmatrix} 1 & 0 & r_1\cos\theta_2 + ir_2\sin\theta \\ & 1 & r_1\sin\theta_2 + ir_2\cos\theta \\ & & r_1^2 + r_2^2 \end{bmatrix}$$

By looking at the firt row of $(T - S)(\mathbf{xx}^*)$, we find that $z_1 = 1, z_2 = 0, \alpha_1 = 0, \alpha_2 = i$ since (1,1)-element is zero. Also, from $S(\mathbf{xx}^*) \geq 0$, we find $\xi_i^*\xi_i \leq 1, \eta_i^*\eta_i \leq 1,\ i = 1, 2$ and hence $w_1 = \beta_2 = 0$. Therefore, we have

$$S(\mathbf{xx}^*) = \begin{bmatrix} 1 & 0 & r_1\cos\theta_2 + ir_2\sin\theta \\ & d_2 & r_1 w_2\sin\theta_2 + r_2\beta_1\cos\theta \\ & & ar_1^2 + br_2^2 + r_1 r_2(f\cos\theta_1 + g\sin\theta_1) \end{bmatrix}.$$

Now, we look at the relation

$$0 \leq \begin{bmatrix} d_2 & r_1 w_2\sin\theta_2 + r_2\cos\theta \\ & ar_1^2 + br_2^2 + r_1 r_2(f\cos\theta_1 + g\sin\theta_1) \end{bmatrix}$$

$$\leq \begin{bmatrix} 1 & r_1\sin\theta_2 + r_2 i\cos\theta \\ & r_1^2 + r_2^2 \end{bmatrix}.$$

Substituting $r_1 = 1, r_2 = 0,\ \theta_2 = \frac{\pi}{2}$, we find that

$$0 \leq \begin{bmatrix} d_2 & w_2 \\ & a \end{bmatrix} \leq \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

and hence $d_2 = w_2 = a$, Similarly, with $r_1 = 0, r_2 = 1, \theta = 0$, we obtain $d_2 = b, \beta_1 = bi$. Thus, we have

$$\begin{aligned} 0 &\leq \begin{bmatrix} a & a(r_1 \sin\theta_2 + ir_2\beta_1 \cos\theta) \\ & a(r_1^2 + r_2^2) + r_1 r_2 (f\cos\theta_1 + g\sin\theta_1) \end{bmatrix} \\ &\leq \begin{bmatrix} 1 & r_1 \sin\theta_2 + ir_2 \cos\theta \\ & r_1^2 + r_2^2 \end{bmatrix} \end{aligned}$$

from which we conclude $f = g = 0$. Finally, we now have

$$S(\mathbf{xx}^*) = \begin{bmatrix} 1 & 0 & r_1 \cos\theta_2 + ir_2 \sin\theta \\ & a & a(r_1 \sin\theta_2 + ir_2 \cos\theta) \\ & & a(r_1^2 + r_2^2) \end{bmatrix}.$$

But $S(\mathbf{xx}^*) \geq 0$ for all $r_1, r_2 \geq 0, \theta, \theta_2 \in \mathbf{R}$ and hence we must have $a \geq 1$, i.e. $a = 1$. Therefore, $S = T$.

LEMMA 3.4. *Let*

$$S(\mathbf{xx}^*) = \begin{bmatrix} r_3^2 I & r_1 r_3 (\mathbf{u}_1 cos\,\theta_2 + \mathbf{u}_2 sin\,\theta_2) + ir_2 r_3 (\mathbf{u}_2 cos\,\theta + \mathbf{u}_1 sin\,\theta) \\ & r_1^2 + r_2^2 \end{bmatrix}$$

with $u_1^T = (cos\,\tau, sin\,\tau), u_2^T = (-sin\,\tau, cos\,\tau)$ for some $\tau \in \mathbf{R}$, then there exists a unitary operator W such that $S \circ W = T$.

Proof. Let $W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i2\tau} & 0 \\ 0 & 0 & e^{i\tau} \end{bmatrix}$ then

$$W \begin{bmatrix} a & \alpha & \beta \\ & b & d \\ & & c \end{bmatrix} W^* = \begin{bmatrix} a & \alpha e{-}2i\tau & \beta i^{-i\tau} \\ & b & \gamma e^{i\tau} \\ & & c \end{bmatrix}$$

and hence we have

$$S(E_{13}) = T(\cos\tau E_{13} - \sin\tau \tilde{E}_{13}) = \begin{bmatrix} 0 & \cos\tau \mathbf{u}_1 - \sin\tau \mathbf{u}_2 \\ & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{e}_1 \\ & 0 \end{bmatrix}.$$

Similarly, we get

$$S(\tilde{E}_{13}) = \begin{bmatrix} 0 & \mathbf{e}_2 \\ & 0 \end{bmatrix},\ S(E_{23}) = \begin{bmatrix} 0 & i\mathbf{e}_2 \\ & 0 \end{bmatrix},$$
$$S(\tilde{E}_{23}) = \begin{bmatrix} 0 & i\mathbf{e}_1 \\ & 0 \end{bmatrix},\ S(E_{12}) = S(\tilde{E}_{12}) = 0.$$

Therefore, S is of the desired form.

LEMMA 3.5. *Let*

$$S(\mathbf{x}\mathbf{x}^*) = \begin{bmatrix} r_3^2 P & r_1 r_3(\xi_1 \cos\theta_2 + \xi_2 \sin\theta_2) + r_2 r_3(\eta_1 \cos\theta + \eta_2 \sin\theta) \\ & a_1 r_1^2 + b r_2^2 + r_1 r_2 (f\cos\theta_1 + g\sin\theta_1) \end{bmatrix}.$$

If $0 \le S \le T$ *then* $P = \lambda I$ *for some* $0 \le \lambda \le 1$.

Proof. Since $P \ge 0$, we can take an orthonormal set of eigenvectors of P. We may take $\mathbf{u}_1, \mathbf{u}_2$ such that $\mathbf{u}_1^T = (\cos\tau, e^{i\lambda}\sin\tau)$, $\mathbf{u}_2^T = (-\sin\tau, e^{i\mu}\cos\tau)$. From $\mathbf{u}_1^*\mathbf{u}_2 = 0$, we have $e^{i\lambda} = e^{i\mu}$. Let $U_0 = (\mathbf{u}_1, \mathbf{u}_2)$, $U = \begin{bmatrix} U_0 & 0 \\ & 1 \end{bmatrix}$, then we have

$$U \circ T(\mathbf{x}\mathbf{x}^*) = \begin{bmatrix} r_3^2 I & r_1 r_3(\mathbf{u}_1 \cos\theta_2 + \mathbf{u}_2 \sin\theta) + i r_2 r_3(\mathbf{u}_2 \cos\theta + \mathbf{u}_1 \sin\theta) \\ & r_1^2 + r_2^2 \end{bmatrix}$$
$$U \circ S(\mathbf{x}\mathbf{x}^*) = \begin{bmatrix} r_3^2 D & r_1 r_3(\xi_1' \cos\theta_2 + \xi_2' \sin\theta_2) + r_2 r_3(\eta_1' \cos\theta + \eta_2' \sin\theta) \\ & a r_1^2 + b r_2^2 + r_1 r_2 (f\cos\theta_1 + g\sin\theta_1) \end{bmatrix}.$$

Let $V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\lambda} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then

$$V \circ U \circ T(\mathbf{x}\mathbf{x}^*) = \begin{bmatrix} r_3^2 I & r_1 r_3(\mathbf{u}_1' \cos\theta_2 + \mathbf{u}_2' \sin\theta_2) + i r_2 r_3(\mathbf{u}_2' \cos\theta + \mathbf{u}_1' \sin\theta) \\ & r_1^2 + r_2^2 \end{bmatrix}$$

where $\mathbf{u}_1'^T = (\cos\tau, \sin\tau)$, $\mathbf{u}_2'^T = (-\sin\tau, \cos\tau)$, and $V \circ U \circ S(\mathbf{x}\mathbf{x}^*) =$

$$\begin{bmatrix} r_3^2 D & r_1r_3(\xi_1''\cos\theta_2 + \xi_2''\sin\theta_2) + r_2r_3(\eta_1''\cos\theta + \eta_2''\sin\theta) \\ & ar_1^2 + br_2^2 + r_1r_2(f'\cos\theta_1 + g'\sin\theta_1) \end{bmatrix}$$

Now, we apply Lemma 3.4 to $V \circ U \circ T$ to find a unitary W such that $V \circ U \circ T \circ W = T$. Then with $S' = V \circ U \circ S \circ W$, we have $0 \le S' \le T$ and

$$S'(\mathbf{x}\mathbf{x}^*) = \begin{bmatrix} r_3^2 D & r_1r_3(\xi_1'''\cos\theta_2 + \xi_2'''\sin\theta_2) + r_2r_3(\eta_1'''\cos\theta + \eta_2'''\sin\theta) \\ & ar_1^2 + br_2^2 + r_1r_2(f''\cos\theta_1 + g''\sin\theta_1) \end{bmatrix}$$

Finally we apply Lemma 3.3 to conclude $D = \lambda I$. Therefore, $P = U_0^* D U_0 = \lambda U_0^* U_0 = \lambda I$.

LEMMA 3.6. *Let*

$$S(\mathbf{x}\mathbf{x}^*) = \begin{bmatrix} dr_3^2 I & r_1r_3(\xi_1\cos\theta_2 + \xi_2\sin\theta_2) + r_2r_3(\eta_1\cos\theta + \eta_2\sin\theta) \\ & ar_1^2 + br_2^2 + r_1r_2(f\cos\theta_1 + g\sin\theta_1) \end{bmatrix}.$$

If $0 \le S \le T$, *then* $S = dT$.

Proof. We apply Lemma 3.1 to $T - S$ where $0 \le T - S \le T$ to obtain

(1) $(\mathbf{e}_1 - \xi_1)^*(\mathbf{e}_1 - \xi_1) = (\mathbf{e}_2 - \xi_2)^*(\mathbf{e}_2 - \xi_2) = (1-a)(1-d)$
(2) $(i\mathbf{e}_2 - \eta_1)^*(i\mathbf{e}_2 - \eta_1) = (i\mathbf{e}_1 - \eta_2)^*(i\mathbf{e} - \eta_2) = (1-b)(1-d)$
(3) $(\mathbf{e}_1 - \xi_1)^*(i\mathbf{e}_2 - \eta_1) + (i\mathbf{e}_2 - \eta_1)^*(\mathbf{e}_1 - \xi_1) = -f(1-d)$
$(\mathbf{e}_2 - \xi_2)^*(i\mathbf{e}_1 - \eta_2) + (i\mathbf{e}_1 - \eta_2)^*(\mathbf{e}_2 - \xi_2) = -f(1-d)$
(4) $(\mathbf{e}_1 - \xi_1)^*(i\mathbf{e}_1 - \eta_2) + (i\mathbf{e}_1 - \eta_2)^*(\mathbf{e}_1 - \xi_1) = g(1-d)$
$(\mathbf{e}_2 - \xi_2)^*(i\mathbf{e}_2 - \eta_1) + (i\mathbf{e}_2 - \eta_1)^*(\mathbf{e}_2 - \xi_2) = -g(1-d)$.

We expand (1) and apply Lemma 3.1 to S so that we have $\mathbf{e}_1^*\xi_1 + \xi_1^*\mathbf{e}_1 = a + d$, $\mathbf{e}_2^*\xi_2 + \xi_2^*\mathbf{e}_2 = a + d$. Thus, we have $\frac{1}{2}(a+d) = Re(\xi_1^*\mathbf{e}_1) \le |\xi_1^*\mathbf{e}_1| \le |\xi_1| = \sqrt{ad}$. Therefore, we obtain $a = d, Re(\xi_1^*\mathbf{e}_1) = |\xi_1| = d$, i.e. $\frac{\xi_1}{|\xi_1|} = \mathbf{e}_1$. From the second relation of (1), we obtain similarly that $\frac{\xi_2}{|\xi_2|} = \mathbf{e}_2$.

We repeat the same process on (2) to obtain $b = d, \mathbf{e}_1 = \frac{-i\eta_2}{|\eta_2|}, \mathbf{e}_2 = \frac{-i\eta_1}{|\eta_1|}$. Using these relations, we find from (3) that $f = 0$ since $\eta_1^*\eta_2 = 0$. Similarly, from the second relation of (4), we have $\frac{i}{|\eta_1|}(\eta_1^*\eta_1 - \eta_1^*\eta_1) = -g$ where we have used the relation $\xi_2^*\eta_1 + \eta_1^*\xi_2 = gd$. Thus, $g = 0$ and hence $S = dT$.

THEOREM 3.7. *T is an extreme poitive linear operator.*

Proof. Let S be an arbitrary positive linear operator with $0 \le S \le T$. By Lemma 2.1, S is of the form

$$S(\mathbf{xx}^*) = \begin{bmatrix} r_3^2 P & r_1 r_3(\xi_1 \cos\theta_2 + \xi_2 \sin\theta_2) + r_2 r_3(\eta_1 \cos\theta + \eta_2 \sin\theta) \\ & a r_1^2 + b r_2^2 + r_1 r_2 (f \cos\theta_1 + g \sin\theta_1) \end{bmatrix}$$

Now, by Lemma 3.5, $P = \lambda I$ for some $0 \le \lambda \le 1$. Therefore, by Lemma 3.6, $S = \lambda T$.

4. Examples of Non-Extreme Positive Linear Operators

EXAMPLE 4.1. Let

$$S(\mathbf{xx}^*) = \begin{bmatrix} r_3^2 I & r_1 r_3 \cos\theta_2 - i r_2 r_2 \sin\theta \\ & r_1 r_3 \sin\theta_2 + i r_2 r_3 \cos\theta \\ & r_1^2 + r_2^2 \end{bmatrix},$$

then S is not extreme.

Proof. We difine

$$S_1(\mathbf{xx}^*) = \begin{bmatrix} r_3^2 & i r_3^2 & r_3(r_1 e^{i\theta_2} - r_2 e^{i\theta}) \\ & r_3^2 & -i r_3(r_1 e^{i\theta_2} - r_2 e^{i\theta}) \\ & & r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta_1 \end{bmatrix}$$

$$S_2(\mathbf{xx}^*) = \begin{bmatrix} r_3^2 & -i r_3^2 & r_3(r_1 e^{i\theta_2} + r_2^{-i\theta}) \\ & r_3^2 & i r_3(r_1 e^{-i\theta_2} + r_2 e^{-i\theta}) \\ & & r_1^2 + r_2^2 + 2 r_1 r_2 \cos\theta_1 \end{bmatrix}$$

then we clearly have $S_1, S_2 \ge 0, S = \frac{1}{2}(S_1 + S_2)$.

EXAMPLE 4.2. Let $S(\mathbf{xx}^*) = \begin{bmatrix} r_1^2 + r_2^2 & 0 & r_1 r_3 \cos\theta_2 + i r_2 r_3 \sin\theta \\ & r_1^2 + r_2^2 & r_1 r_3 \sin\theta_2 + i r_2 r_3 \cos\theta \\ & & r_3^2 \end{bmatrix}$,

then S is not extreme.

Proof. We define $S_1(\mathbf{xx}^*) =$

$$\begin{bmatrix} r_1^2 + r_2^2 + 2 r_1 r_2 \cos\theta_1 & i(r_1^2 - r_2^2) + 2 r_1 r_2 \sin\theta_1 & r_1 r_3 e^{i\theta_2} + r_2 r_3 e^{i\theta} \\ & r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta_1 & -i(r_1 r_3 e^{i\theta_2} - r_2 r_3 e^{i\theta}) \\ & & r_3^2 \end{bmatrix},$$

and $S_2(\mathbf{xx}^*) =$

$$\begin{bmatrix} r_1^2+r_2^2-2r_1r_2\cos\,\theta_1 & -i(r_1^2-r_2^2)-2r_1r_2\sin\,\theta_1 & r_1r_3e^{-i\theta_2}-r_2r_3e^{-i\theta} \\ & r_1^2+r_2^2+2r_1r_2\cos\,\theta_1 & i(r_1r_3e^{-i\theta_2}-r_2r_3e^{-i\theta}) \\ & & r_3^2 \end{bmatrix}$$

then it is routine to verify that $S_1, S_2 \geq 0$ and $S = \frac{1}{2}S_1 + \frac{1}{2}S_2$.

EXAMPLE 4.3. Let $S(\mathbf{xx}^*) = \begin{bmatrix} r_3^2 I & r_1r_3\cos\,\theta_2 + ir_2r_3\sin\,\theta \\ & ir_1r_3\sin\,\theta_2 + r_2r_3\cos\,\theta \\ & r_1^2+r_2^2 \end{bmatrix}$, then S is not extreme.

Proof. We define

$$S_1(\mathbf{xx}^*) = \begin{bmatrix} r_3^2 & r_3^2 & r_1e^{i\theta_2}+r_2e^{i\theta} \\ & r_3^2 & r_1e^{i\theta_2}+r_2e^{i\theta} \\ & & r_1^2+r_2^2+2r_1r_2\cos\,\theta_1 \end{bmatrix},$$

$$S_2(\mathbf{xx}^*) = \begin{bmatrix} r_3^2 & -r_3^2 & r_1r_3e^{-i\theta_2}-r_2r_3e^{-i\theta} \\ & r_3^2 & -r_1r_3e^{-i\theta_2}+r_2r_3e^{-i\theta} \\ & & r_1^2+r_2^2-2r_1r_2\cos\,\theta_1 \end{bmatrix}$$

then we clealy have $S_1, S_2 \geq 0, S = \frac{1}{2}S_1 + \frac{1}{2}S_2$.

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Korea Atomic Energy Research Institute
Taejon 305–353, Korea