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$\mathbb{Z}_p$–LOCALIZATION IN THE PLUS–CONSTRUCTION AND ITS APPLICATIONS

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1. Introduction

Throughout this paper, A means a ring with unity. It is well-known that $BGLA^+$ is an infinite loop space, a nilpotent space, and a homotopy associative H-space [1, 2, 6, 10 12, 14, 16]. Since D. Quillen defined the algebraic K-group $K_n(A) = \pi_n(BGLA^+)$ for $n \geq 1$, $K_n(A)$ is an abelian group.

In this paper, we shall prove that $R_\infty BGLA^+$ is an infinite loop space (Theorem 3.1) and $[X,\ R_\infty BGLA^+]$, $[X, BGLA^+]$ are trivial groups where $[\,,]$ means the set of based homotopy classes, X is an acyclic space (Corollary 3.2). Finally, by use of the Ext-completion, Hom-completion [3,4] and Dror's acyclic tower, we shall calculate the homotopy group $\pi_n(R_\infty BGLF_q^+)$ where F_q is a finite field with q elements. More precisely,

$$\pi_n(R_\infty BGLF_q^+) \cong \begin{cases} 0 & \text{if } n \text{ is even }, \\ \underline{\underline{\mathbb{Z}}}_p \otimes \mathbb{Z}_{q^j-1} & \text{if } n(=2j-1) \text{ is odd}, \end{cases}$$

where $R = \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ and R_∞ means the Bousfield's R-completion which is equivalent to the R-localization in the S_{*N}.

Throughout this paper, we shall work in the category of based connected CW-complexes which is denoted by S_*. Moreover we denote the category of based connected nilpotent CW-complexes by S_{*N}. Obviously S_{*N} is a subcategory of S_*. All maps means the base point preserving continuous maps unless otherwise stated. We denote the maximal perfect subgroup of a group G by PG, and the loop functor by Ω.

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2. Preliminaries

In this section, we shall consider the Dror's acyclic fiber and Quillen's plus-construction, nilpotent action, the localization of a nilpotent space and the completion of a space.

Firstly recall the Quillen's plus-construction; let X be a space then the Quillen's plus-construction X^+[15] is constructed as follows. Let $p : X' \to X$ be a covering space of X with $\pi_1(X') = P\pi_1(X)$. Attach 2-cells and 3-cells to X' to get a simply connected space Y' such that $f : X' \to Y'$ is an acyclic cofibration. Then we have the acyclic cofibration $i^+ : X \to X^+$ by the push-out with $\ker \pi_1(i^+) = P\pi_1(X)$.

Secondly recall the infinite general linear group $GLA = \varinjlim GL_n A$ where A is the ring with unity. We consider the $BGLA$ as the Milnor classifying space of GLA and $BGLA^+$ as the plus-construction of the $BGLA$.

Now, we recall thirdly the Dror's acyclic fiber over X[5]. We take $X_1 = X$ and X_2 as the covering space of X with $\pi_1(X_2) = P\pi_1(X)$. Next $X_3, X_4, \ldots$ are construct as follows;

$$X_2 \leftarrow X_3 \leftarrow X_4 \leftarrow \cdots \leftarrow X_n \leftarrow X_{n+1} \leftarrow \cdots$$

such that

(1) $H_q(X_n) = 0,\ q < n.$

(2) $X_n \leftarrow X_{n+1}$ is induced from the path fibration $(n \geq 2)$

$$\begin{array}{ccc} X_{n+1} & \longrightarrow & \Lambda K(H_n(X_n), n) \\ \downarrow & © & \downarrow \\ X_n & \longrightarrow & K(H_n(X_n), n) \end{array} \quad ;\text{cartesian square (pull-back)}$$

where Λ means the path space, K means the Eilenberg-MacLane space.

(3) X_n is unique up to fiber homotopy equivalence over X_{n-1}.

Then $\varprojlim X_n = AX$ is an acyclic space.

LEMMA 2.1.

$$\pi_q(X_n^+) = \begin{cases} 0 & \text{if } q < n, \\ \pi_q(X^+) & \text{if } q \geq n. \end{cases}$$

Proof. We consider the fibration $X_2 \to X \to K(\pi_1(X^+), 1)$. Since $P\pi_1(X^+)$ is a trivial group, there exists a fibration;

$$X_2^+ \to X^+ \to K(\pi_1(X^+), 1)^+ = K(\pi_1(X^+), 1).$$

As the same method we also make the following fibration;

$$X_{n+1}^+ \to X_n^+ \to K(H_n(X_n), n)^+ = K(H_n(X_n), n) \quad \text{for } n \geq 2.$$

By use of the homotopy exact sequence and generalized Hurewicz theorem, we have

$$\begin{cases} \pi_q(X_n^+) = 0 & \text{if } q < n, \\ \pi_q(X_n^+) = \pi_q(X^+) & \text{if } q \geq n. \end{cases}$$

DEFINITION 2.2. A group acts *nilpotently* on a group G if there is an action on G with the following conditions; there exists a finite sequence of subgroups of G i.e.,

$$G = G_1 \supset G_2 \supset \cdots \supset G_j \supset \cdots \supset G_n = *$$

such that for all j

(1) G_j is closed under the action,
(2) G_{j+1} is normal subgroup of G_j, G_j/G_{j+1} is abelian,
(3) The induced action on G_j/G_{j+1} is trivial.

Next X is called a *nilpotent space* if

(1) $\pi_1(X)$ is a nilpotent group,
(2) The action $\pi_1(X)$ on $\pi_n(X)$ is nilpotent for $n \geq 2$.

DEFINITION 2.3. For $X \in S_{*N}$, $R \subset \mathbb{Q}$, an R*-localization* of X is the space $\bar{X}$ with the map $X \to \bar{X} \in S_{*N}$ such that either of the following (equivalent) conditions hold;

(1) the groups $\pi_*\bar{X}$ are R-nilpotent and the canonical map

$$R \otimes \pi_* X \to \pi_* \bar{X} \text{ is an isomorphism,}$$

(2) the groups $H_*(\bar{X}; \mathbb{Z})$ are R-nilpotent and the canonical map

$$R \otimes \tilde{H}_*(X : \mathbb{Z}) \to \tilde{H}_*(\bar{X}; \mathbb{Z}) \text{ is an isomorphism.}$$

Since the R-completion $X \to R_\infty X$ is an R-localization for $X \in S_{*N}$ [9, 12] and any R-localization $X \to \bar{X}$ is canonically equivalent to $X \to R_\infty X$ in the pointed homotopy category [4]. Moreover the R-completion has the following properties [11];

(1) $R_\infty(SX) \approx SR_\infty X$, where S denotes the suspension functor.
(2) $R_\infty(\Omega X) \approx \Omega R_\infty X$, where Ω denotes the loop functor and X is 1-connected.

3. Main Theorems

In this section, we shall prove that $R_\infty BGLA^+$ is an infinite loop space and show the group structure of $[X, R_\infty BGLA^+]$. Finally, $\pi_n(R_\infty BGLF_q^+)$ is calculated.

We know that $BGLA^+$ is a nilpotent space [10, 14], thus we consider the R-localization and R-completion of the $BGLA^+$.

Let CA be the ring of locally finite matrices over A and $MA(\subset CA)$ be the two-sided ideal of finite matrices, i.e., those matrices have at most finitely many non-zero entries. Define $SA = CA/MA$ which is called the *suspension* ring of A.

THEOREM 3.1. *$R_\infty BGLA^+$ is an infinite loop space.*

Proof. We know that precise Ω-spectrum structure on $BGLA^+$ is shown [2, 16];

$$BGLA^+, BGL(SA)_2^+, \ldots, BGL(S^nA)_{n+1}^+, \ldots$$

where $(\quad)_n$ means the Dror's n-th acyclic tower[5].
By lemma 2.1, $BGL(SA)_2^+$ is a 1-connected space and $BGL(S^2A)_3^+$ is a 2-connected space. Now we make the new sequence;

$$R_\infty BGLA^+, R_\infty BGL(SA)_2^+, \ldots, R_\infty BGL(S^nA)_{n+1}^+, \ldots.$$

In the sequence above,

$$\begin{aligned} R_\infty BGLA^+ &\approx R_\infty \Omega BGL(SA)_2^+ \\ &\approx \Omega R_\infty BGL(SA)_2^+ \end{aligned}$$

because $BGL(SA)_2^+$ is 1-connected.

" $\approx$ " means homotopy equivalence.

Furthermore, for all $n \geq 2$

$$\begin{aligned} R_\infty BGL(S^{n-1}A)_n^+ &\approx R_\infty \Omega BGL(S^n A)_{n+1}^+ \\ &\approx \Omega R_\infty BGL(S^n A)_{n+1}^+ \end{aligned}$$

because $BGL(S^n A)_{n+1}^+$ is an n-connected space. Therefore $R_\infty BGLA^+$ is an infinite loop space.

COROLLARY 3.2. *$[X, R_\infty BGLA^+]$, $[X, BGLA^+]$ are all trivial groups where X is an acyclic space.*

We recall that $K_n(F_q) = \pi_n(BGLF_q^+)$ is a finitely generated abelian group [8, 15]. i.e.,

$$\begin{cases} K_{2j}(F_q) = 0 \\ K_{2j-1}(F_q) \cong \mathbb{Z}_{q^j-1}. \end{cases}$$

Next, for every nilpotent group N and prime number p we recall the Ext-completion;

$$\mathrm{Ext}(\mathbb{Z}_{p^\infty}, N) = \pi_1 R_\infty K(N, 1)$$

and Hom-completion;

$$\mathrm{Hom}(\mathbb{Z}_{p^\infty}, N) = \pi_2 R_\infty K(N, 1)$$

where R_∞ means the Bousfield's R-completion which is equivalent to the R-localization in S_{*N} $R = \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ [7, 9, 12].

THEOREM 3.3. *If F_q is a finite field with q-elements, then*

$$\pi_n(R_\infty BGLF_q^+) \cong \begin{cases} 0 & \text{if } n \text{ is even}, \\ \underline{\mathbb{Z}_p} \otimes \mathbb{Z}_{q^j-1} & \text{if } n(=2j-1) \text{ is odd}, \end{cases}$$

where $\underline{\mathbb{Z}_p} = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is a p-adic integers.

Proof. For $X \in S_{*N}$ and $R = \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, we know that $R_\infty X \in S_{*N}$. Now, we consider the following splittable short exact sequence;[4]

$$0 \to \mathrm{Ext}(\mathbb{Z}_{p^\infty}, \pi_n X) \to \pi_n(R_\infty X) \to \mathrm{Hom}(\mathbb{Z}_{p^\infty}, \pi_{n-1} X) \to 0.$$

Since $BGLA^+$ is a nilpotent space, we can make the following sequence;

$$0 \to \operatorname{Ext}(\mathbb{Z}_{p^\infty}, \pi_n BGLF_q^+) \to \pi_n(R_\infty BGLF_q^+) \\ \to \operatorname{Hom}(\mathbb{Z}_{p^\infty}, \pi_{n-1} BGLF_q^+) \to 0.$$

Thus we have

$$\pi_n(R_\infty BGLF_q^+) \cong \operatorname{Ext}(\mathbb{Z}_{p^\infty}, \pi_n BGLF_q^+) \oplus \operatorname{Hom}(\mathbb{Z}_{p^\infty}, \pi_{n-1} BGLF_q^+)$$

for every $n \geq 1$. And $\pi_{n-1} BGLF_q^+$ is a finitely generated abelian group. Hence

$$\operatorname{Hom}(\mathbb{Z}_{p^\infty}, \pi_{n-1} BGLF_q^+) = 0.$$

Furthermore

$$\operatorname{Ext}(\mathbb{Z}_{p^\infty}, \pi_n BGLF_q^+) \cong \underline{\underline{\mathbb{Z}}}_p \otimes K_n(F_q).$$

Therefore

$$\pi_n(R_\infty BGLF_q^+) \cong \begin{cases} 0 & \text{if } n \text{ is even ,} \\ \underline{\underline{\mathbb{Z}}}_p \otimes \mathbb{Z}_{q^j-1} & \text{if } n(=2j-1) \text{ is odd .} \end{cases}$$

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