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INDEFINITE KAEHLERIAN MANIFOLDS WITH PARALLEL CONFORMAL CURVATURE TENSOR FIELD

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1. Introduction

S. Bochner[3] has introduced "Bochner curvature tensor" on a Kaehlerian manifold analogous to the Weyl conformal curvature tensor on a Riemannian manifold. In 1990, H. Kitahara, K. Matsuo and J. S. Pak [6],[7] defined a new tensor field on a Hermitian manifold which is conformally invariant and studied several properties of the new tensor field. They called this new tensor field "conformal curvature tensor field". On the other hand, in 1987, R. Aiyama, J.-H. Kwon and H. Nakagawa [1] studied several properties of indefinite Kaehlerian manifold. Recently, J.-H. Kwon and W.-H. Sohn [9] investigated some properties of locally product indefinite Kaehlerian metrics with vanishing conformal curvature tensor field.

The purpose of this paper is to study indefinite Kaehlerian manifolds with parallel or vanishing conformal curvature tensor field. In the second section, a brief summary of the complex version of indefinite Kaehlerian manifolds is recalled and some fundamental formulas of indefinite complex submanifolds of an indefinite Kaehlerian manifold are prepared. Section 3 is devoted to the investigation of some properties of indefinite Kaehlerian manifold with parallel or vanishing conformal curvature tensor field.

2. Indefinite Kaehlerian manifolds

We start this section by introducing some basic formulas concerning indefinite Kaehlerian manifolds. Let M be a complex $n(\geq 2)$-dimensional

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connected indefinite Kaehlerian manifold equipped with Kaehlerian metric tensor g and almost complex structure J. For the indefinite Kaehlerian structure (g, J), we know that J is integrable and the index of g is even, say $2s(0 \leq s \leq n)$.

A local unitary frame field $\{E_1, \ldots, E_n\}$ on a neighborhood of M can be chosen. This is a complex linear frame which is orthonormal with respect to the Kaehlerian metric, that is, $(E_i, \overline{E}_j) = \varepsilon_i \delta_{ij}$, where $\varepsilon_i = \pm 1$. The dual frame field $\{\omega_1, \ldots, \omega_n\}$ of the frame field $\{E_j\}$ consists of complex-valued 1-forms ω_i of type $(1,0)$ on M such that $\omega_i(E_j) = \varepsilon_i \delta_{ij}$ and $\{\omega_1, \ldots, \omega_n, \overline{\omega}_1, \ldots, \overline{\omega}_n\}$ is linearly independent. The Kaehlerian metric g of M can be expressed as $g = 2\sum_j \varepsilon_j \omega_j \otimes \overline{\omega}_j$. Associated with the frame field $\{E_j\}$, there exist complex-valued 1-forms ω_{ij}, which are usually *connection forms* on M such that they satisfy the structure equations of M:

$$
\begin{aligned}
&d\omega_i + \sum_j \varepsilon_j \omega_{ij} \wedge \omega_j = 0, \ \ \omega_{ij} + \overline{\omega}_{ji} = 0, \\
&d\omega_{ij} + \sum_k \varepsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\
&\Omega_{ij} = \sum_{k,l} \varepsilon_k \varepsilon_l R_{\bar{i}jk\bar{l}} \omega_k \wedge \overline{\omega}_l,
\end{aligned}
\tag{2.1}
$$

where Ω_{ij}(resp. $R_{\bar{i}jk\bar{l}}$) denotes the curvature form (resp. the components of the Riemannian curvature tensor R) on M. The second equation of (2.1) means the skew-hermitian symmetry of Ω_{ij}, which is equivalent to the symmetric condition

$$R_{\bar{i}jk\bar{l}} = \overline{R}_{\bar{j}il\bar{k}}. \tag{2.2}$$

The Bianchi identity obtained by the exterior derivatives of (2.1) gives $\sum_j \varepsilon_j \Omega_{ij} \wedge \omega_j = 0$, which yields the following further symmetric relations

$$R_{\bar{i}jk\bar{l}} = R_{\bar{i}kj\bar{l}} = R_{\bar{l}jk\bar{i}} = R_{\bar{l}kj\bar{i}}. \tag{2.3}$$

Now, relative to the frame field chosen above, the Ricci tensor S of M can be expressed as follows:

$$S = 2\sum_{i,j} \varepsilon_i \varepsilon_j S_{i\bar{j}} \omega_i \otimes \overline{\omega}_j,$$

where $S_{i\bar{j}} = \sum_k \varepsilon_k R_{\bar{k}ki\bar{j}} = S_{\bar{j}i} = \overline{S}_{\bar{i}j}$. The scalar curvature r is also given by $r = 2\sum_j \varepsilon_j S_{j\bar{j}}$. The indefinite Kaehlerian manifold M is said to be *Einstein* if the Ricci tensor S is given by

$$S_{\bar{j}i} = \alpha\varepsilon_i\delta_{ij}, \tag{2.4}$$

where $\alpha = \frac{r}{2n}$. The components $R_{\bar{i}jk\bar{l}m}$ and $R_{\bar{i}jk\bar{l}\bar{m}}$ (resp. $S_{\bar{j}ik}$ and $S_{\bar{j}i\bar{k}}$) of the covariant derivative of the Riemannian curvature tensor R (resp. the Ricci tensor S) are defined by the following equation (2.5) (resp. (2.6))

$$\begin{aligned}(2.5)\quad \sum_m \varepsilon_m(R_{\bar{i}jk\bar{l}m}\omega_m + R_{\bar{i}jk\bar{l}\bar{m}}\overline{\omega}_m) &= dR_{\bar{i}jk\bar{l}} \\ &- \sum_m \varepsilon_m(R_{\overline{m}jk\bar{l}}\overline{\omega}_{mi} + R_{\bar{i}mk\bar{l}}\omega_{mj} + R_{\bar{i}jm\bar{l}}\omega_{mk} + R_{\bar{i}jk\overline{m}}\overline{\omega}_{ml}),\end{aligned}$$

$$\sum_k \varepsilon_k(S_{\bar{j}ik}\omega_k + S_{\bar{j}i\bar{k}}\overline{\omega}_k) = dS_{\bar{j}i} - \sum_k \varepsilon_k(S_{\bar{j}k}\omega_{ki} + S_{\bar{k}i}\overline{\omega}_{kj}). \tag{2.6}$$

The second Bianchi formula is given by $R_{\bar{i}jk\bar{l}m} = R_{\bar{i}jm\bar{l}k}$ and hence we have

$$S_{\bar{j}ik} = S_{\bar{j}ki} = \sum_l \varepsilon_l R_{\bar{j}ik\bar{l}l}, \quad r_j = 2\sum_k \varepsilon_k S_{\bar{k}jk}, \tag{2.7}$$

where $dr = \sum_j \varepsilon_j(r_j\omega_j + r_{\bar{j}}\overline{\omega}_j)$.

Now let M' be an $(n+p)$-dimensional connected Kaehlerian manifold of index $2(s+t)(0 \leq s \leq n,\ 0 \leq t \leq p)$ and let M be an n-dimensional connected indefinite complex submanifold of M' of index $2s$. Then M is the indefinite Kaehlerian manifold endowed with the induced metric tensor g. We choose a local unitary frame field $\{E_A\} = \{E_1, \ldots, E_{n+p}\}$ on a neighborhood of M' in such a way that restricted to M, $E_1, \ldots, E_n$ are tangent to M and the others are normal to M. Here and in the sequel the following convention on the range of indices is used unless otherwise stated:

$$A,\ B,\ C, \ldots = 1, \ldots,\ n,\ n+1, \ldots,\ n+p,$$

$$i,\ j,\ k, \ldots = 1, \ldots,\ n,$$

$$x,\ y,\ z, \ldots = n+1, \ldots,\ n+p.$$

With respect to the frame field, let $\{\omega_A\} = \{\omega_i, \omega_x\}$ be its dual frame field. Then the Kaehlerian metric tensor g' of M' is given by $g' = 2\sum_A \varepsilon_A \omega_A \otimes \overline{\omega}_A$. The connection forms on M' are denoted by ω_{AB}. The canonical forms ω_A and the connection forms ω_{AB} of the ambient space satisfy the structure equations (2.1).

Restricting these forms to the submanifold M, we have $\omega_x = 0$ and the induced indefinite Kaehlerian metric tensor g of index $2s$ of M is given by $g = 2\sum_j \varepsilon_j \omega_j \otimes \overline{\omega}_j$. Then $\{E_j\}$ is a local unitary frame field with respect to this metric and $\{\omega_j\}$ is a local dual frame field due to $\{E_j\}$ which consists of complex-valued 1-forms of type $(1,0)$ on M. Moreover $\{\omega_1, \ldots, \omega_n, \overline{\omega}_1, \ldots, \overline{\omega}_n\}$ is linearly independent and they are canonical forms on M. It follows from $\omega_x = 0$ and the Cartan lemma that the exterior derivatives of $\omega_x = 0$ give rise to

$$\omega_{xi} = \sum_j \varepsilon_j h^x_{ij} \omega_j,\ h^x_{ij} = h^x_{ji}. \tag{2.8}$$

The quadratic form $\sum_{i,j,x} \varepsilon_i \varepsilon_j \varepsilon_x h^x_{ij} \omega_i \otimes \omega_j \otimes E_x$ with values in the normal bundle is called the *second fundamental form* of the submanifold M. From the structure equations of M' it follows that the structure equations of M are similarly given by (2.1). Moreover the following relationships are defined:

$$\begin{aligned} &d\omega_{xy} + \sum_z \varepsilon_z \omega_{xz} \wedge \omega_{zy} = \Omega_{xy}, \\ &\Omega_{xy} = \sum_{k,l} \varepsilon_k \varepsilon_l R_{\bar{x}yk\bar{l}} \omega_k \wedge \overline{\omega}_l, \end{aligned} \tag{2.9}$$

where Ω_{xy} is called the *normal curvature form* of M. For the Riemannian curvature tensors R and R' of M and M' respectively, it follows from the third equation of (2.1) and (2.8) that the Gauss equation

$$R_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} - \sum_x \varepsilon_x h^x_{jk} \overline{h}^x_{il} \tag{2.10}$$

holds and from (2.2), (2.8) and (2.9) that

$$R_{\bar{x}yk\bar{l}} = R'_{\bar{x}yk\bar{l}} + \sum_j \varepsilon_j h^x_{kj} \overline{h}^y_{jl}$$

holds too. The components of the Ricci tensor S and the scalar curvature r of M are given by

$$S_{\bar{j}i} = \sum_k \varepsilon_k R'_{\bar{k}ki\bar{j}} - (h_{\bar{j}i})^2, \tag{2.11}$$

$$r = 2\sum_{j,k} \varepsilon_j \varepsilon_k R'_{\bar{j}jk\bar{k}} - 2h_2, \tag{2.12}$$

where $(h_{\bar{j}i})^2 = \sum_{r,x} \varepsilon_r \varepsilon_x h^x_{ir} \overline{h}^x_{rj}$ and $h_2 = \sum_i \varepsilon_i (h_{\bar{i}i})^2$.

3. Main results

This section is devoted to the investigation of indefinite Kaehlerian manifolds with parallel or vanishing conformal curvature tensor field.

Let M be a complex n-dimensional indefinite Kaehlerian manifold. The conformal curvature tensor field B_0 with components $B_{0,\bar{i}jk\bar{l}}$ of M is given by

$$\begin{aligned} B_{0,\bar{i}jk\bar{l}} =& R_{\bar{i}jk\bar{l}} - \frac{1}{n}(\varepsilon_j \delta_{ij} S_{\bar{l}k} + \varepsilon_k S_{\bar{i}j} \delta_{kl}) \\ &+ \frac{(n+2)r}{2n^2(n+1)} \varepsilon_j \varepsilon_k \delta_{ij} \delta_{kl} - \frac{r}{2n(n+1)} \varepsilon_j \varepsilon_k \delta_{ik} \delta_{jl}, \end{aligned} \tag{3.1}$$

which was introduced by H. Kitahara, K. Matsuo and J. S. Pak [6].

Let b_0 denote the Ricci contraction of B_0, that is,

$$b_{0,\bar{i}j} = \sum_k \varepsilon_k B_{0,\bar{i}kj\bar{k}}. \tag{3.2}$$

From (3.1) and (3.2), we have

$$b_{0,\bar{i}j} = \frac{n-2}{n}(S_{\bar{i}j} - \frac{r}{2n}\varepsilon_j \delta_{ij}). \tag{3.3}$$

Summing up the equation (3.3) for i and j and using $r = 2\sum_j \varepsilon_j S_{\bar{j}j}$, we obtain

$$(3.4) \qquad \sum_j \varepsilon_j b_{0,\bar{j}j} = 0.$$

If the Ricci contraction b_0 vanishes everywhere i.e. $b_{0,\bar{i}j} = 0$ and $n > 2$, then we obtain $S_{\bar{i}j} = \frac{r}{2n}\varepsilon_j \delta_{ij}$ from (3.3). Since this equation represents the first Chern class, it follows that r is constant. Thus M is Einstein by (2.4). Conversely, if M is Einstein, then we have $b_{0,\bar{i}j} = 0$ by (2.4) and (3.3). Hence we have

LEMMA 3.1. *Let M be an indefinite Kaehlerian manifold of complex dimension $n (n > 2)$. Then the Ricci contraction b_0 of the conformal curvature tensor field B_0 of M vanishes everywhere if and only if M is Einstein.*

On the other hand, the Bochner curvature tensor B with components $B_{\bar{i}jk\bar{l}}$ of the indefinite Kaehlerian manifold is given by

$$(3.5) \quad \begin{aligned} B_{\bar{i}jk\bar{l}} = R_{\bar{i}jk\bar{l}} &- \frac{1}{n+2}(\varepsilon_j \delta_{ij} S_{\bar{l}k} + \varepsilon_k S_{\bar{i}j}\delta_{kl} + \varepsilon_j \delta_{ik} S_{\bar{l}j} + \varepsilon_k S_{\bar{i}k}\delta_{jl}) \\ &+ \frac{r}{2(n+1)(n+2)}\varepsilon_j \varepsilon_k (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}), \end{aligned}$$

which was introduced by S. Bochner [3]. Thus, from (3.1) and (3.5), we have

$$(3.6) \quad \begin{aligned} B_{0,\bar{i}jk\bar{l}} = B_{\bar{i}jk\bar{l}} &+ \frac{1}{n+2}(\varepsilon_k \delta_{ik} S_{\bar{l}j} + \varepsilon_j S_{\bar{i}k}\delta_{jl} - \frac{r}{n}\varepsilon_j \varepsilon_k \delta_{ik}\delta_{jl}) \\ &- \frac{2}{n(n+1)}(\varepsilon_j \delta_{ij} S_{\bar{l}k} + \varepsilon_k S_{\bar{i}j}\delta_{kl} - \frac{r}{n}\varepsilon_j \varepsilon_k \delta_{ij}\delta_{kl}). \end{aligned}$$

If $n > 2$, then from (3.3) and the above equation (3.6), we have

$$(3.7) \quad \begin{aligned} B_{0,\bar{i}jk\bar{l}} = B_{\bar{i}jk\bar{l}} &+ \frac{n}{(n+2)(n-2)}(\varepsilon_k \delta_{ik} b_{0,\bar{l}j} + \varepsilon_j b_{0,\bar{i}k}\delta_{lj}) \\ &- \frac{2}{(n+2)(n-2)}(\varepsilon_j \delta_{ij} b_{0,\bar{l}k} + \varepsilon_k b_{0,\bar{i}j}\delta_{lk}). \end{aligned}$$

Assume that $B_0 = B$ and $n > 2$. Then from (3.7), it follows that

$$n(\varepsilon_k\delta_{ik}b_{0,\bar{l}j} + \varepsilon_j b_{0,\bar{i}k}\delta_{lj}) - 2(\varepsilon_j\delta_{ij}b_{0,\bar{l}k} + \varepsilon_k b_{0,\bar{i}j}\delta_{lk}) = 0.$$

Summing up the above equation for i and k and using (3.4), we get $b_{0,\bar{i}j} = 0$. Conversely, if $b_{0,\bar{i}j} = 0$ and $n > 2$, then from (3.7) we obtain $B_0 = B$. Hence from Lemma 3.1, we have

PROPOSITION 3.2. *Let M be an indefinite Kaehlerian manifold of complex dimension $n(n > 2)$. Then the conformal curvature tensor field is equal to the Bochner curvature tensor on M if and only if M is Einstein.*

REMARK. Let M be an indefinite Kaehlerian manifold of complex dimension 2. If M is Einstein, then the conformal curvature tensor field is equal to the Bochner curvature tensor on M. In fact, substituting (2.4) into (3.6) implies our result.

The components $b_{0,\bar{i}jk}$ and $b_{0,\bar{i}j\bar{k}}$ of the covariant derivative of the Ricci contraction b_0 of the conformal curvature tensor field B_0 are defined by

$$(3.8)\quad \sum_k \varepsilon_k(b_{0,\bar{i}jk}\omega_k + b_{0,\bar{i}j\bar{k}}\overline{\omega}_k) = db_{0,\bar{i}j} - \sum_k \varepsilon_k(b_{0,\bar{k}j}\overline{\omega}_{ki} + b_{0,\bar{i}k}\omega_{kj}).$$

Since $dr = \sum_j \varepsilon_j(r_j\omega_j + r_{\bar{j}}\overline{\omega}_j)$, from (2.1), (2.6), (3.3) and (3.8), we have

$$\begin{aligned}&\sum_k \varepsilon_k(b_{0,\bar{i}jk}\omega_k + b_{0,\bar{i}j\bar{k}}\overline{\omega}_k)\\ &= \frac{n-2}{n}\sum_k \varepsilon_k\{S_{\bar{i}jk}\omega_k + S_{\bar{i}j\bar{k}}\overline{\omega}_k - \frac{1}{2n}\varepsilon_j\delta_{ij}(r_k\omega_k + r_{\bar{k}}\overline{\omega}_k)\},\end{aligned}$$

which yields

$$(3.9)\qquad \begin{aligned} b_{0,\bar{i}jk} &= \frac{n-2}{n}(S_{\bar{i}jk} - \frac{1}{2n}\varepsilon_j\delta_{ij}r_k),\\ b_{0,\bar{i}j\bar{k}} &= \frac{n-2}{n}(S_{\bar{i}j\bar{k}} - \frac{1}{2n}\varepsilon_j\delta_{ij}r_{\bar{k}}).\end{aligned}$$

Assume that the Ricci contraction b_0 is parallel, i.e. $b_{0,\bar{i}jk} = 0$ and $b_{0,\bar{i}j\bar{k}} = 0$. If $n > 2$, then from (3.9) we have

$$S_{\bar{i}jk} = \frac{1}{2n}\varepsilon_j\delta_{ij}r_k, \; S_{\bar{i}j\bar{k}} = \frac{1}{2n}\varepsilon_j\delta_{ij}r_{\bar{k}}. \tag{3.10}$$

From (2.7) and (3.9), we have $r_k = 0$ and $r_{\bar{k}} = 0$. Substituting these equations into (3.10), we obtain $S_{\bar{i}jk} = 0$ and $S_{\bar{i}j\bar{k}} = 0$, that is, the Ricci tensor is parallel.

Conversely, if the Ricci tensor is parallel, then $r_k = 0$ and $r_{\bar{k}} = 0$, and consequently $b_{0,\bar{i}jk} = 0$ and $b_{0,\bar{i}j\bar{k}} = 0$ with the help of (3.9). Thus we have

PROPOSITION 3.3. *Let M be an indefinite Kaehlerian manifold of complex dimension $n(n > 2)$. Then the Ricci contraction of the conformal curvature tensor field is parallel if and only if the Ricci tensor is parallel.*

From Proposition 3.3 and Theorem due to J.-H. Kwon [8], we have the following

COROLLARY 3.4. *Let M be an indefinite Kaehlerian manifold of complex dimension $n(n > 2)$. The following assertions are equivalent*

(1) *the Ricci contraction of the conformal curvature tensor field of M is parallel,*
(2) *M has harmonic curvature,*
(3) *the Ricci tensor of M is cyclic-parallel.*

The components $B_{0,\bar{i}jk\bar{l}m}$ and $B_{0,\bar{i}jk\bar{l}\bar{m}}$ of the covariant derivative of the conformal curvature tensor field B_0 are defined by

$$\begin{aligned}\sum_m \varepsilon_m(B_{0,\bar{i}jk\bar{l}m}\omega_m + B_{0,\bar{i}jk\bar{l}\bar{m}}) = dB_{0,\bar{i}jk\bar{l}} - \sum_m \varepsilon_m(B_{0,\bar{i}jk\bar{l}}\bar{\omega}_{mi} \\ + B_{0,\bar{i}mk\bar{l}}\omega_{mj} + B_{0,\bar{i}jm\bar{l}}\omega_{mk} + B_{0,\bar{i}jk\bar{m}}\bar{\omega}_{ml}).\end{aligned} \tag{3.11}$$

Since $dr = \sum_j \varepsilon_j(r_j\omega_j + r_{\bar{j}}\bar{\omega}_j)$, it follows from (2.1), (2.5), (2.6), (3.1) and (3.4) that

$$\begin{aligned}B_{0,\bar{i}jk\bar{l}m} = R_{\bar{i}jk\bar{l}m} - \frac{1}{n}(\varepsilon_j\delta_{ij}S_{\bar{l}km} + \varepsilon_k S_{\bar{i}jm}\delta_{kl}) \\ + \frac{n+2}{2n^2(n+1)}r_m\varepsilon_j\varepsilon_k\delta_{ij}\delta_{kl} - \frac{1}{2n(n+1)}r_m\varepsilon_j\varepsilon_k\delta_{ik}\delta_{jl},\end{aligned} \tag{3.12}$$

$$(3.13)\ B_{0,\bar{i}jk\bar{l}\bar{m}} = R_{\bar{i}jk\bar{l}\bar{m}} - \frac{1}{n}(\varepsilon_j\delta_{ij}S_{\bar{l}k\bar{m}} + \varepsilon_k S_{\bar{i}j\bar{m}}\delta_{kl}) + \frac{n+2}{2n^2(n+1)}r_{\bar{m}}\varepsilon_j\varepsilon_k\delta_{ij}\delta_{kl} - \frac{1}{2n(n+1)}r_{\bar{m}}\varepsilon_j\varepsilon_k\delta_{ik}\delta_{jl}.$$

If the conformal curvature tensor field of M is parallel, it follows that $b_{0,\bar{i}jk} = 0$ and $b_{0,\bar{i}j\bar{m}} = 0$, that is, the Ricci contraction b_0 is parallel. Thus, by Proposition 3.3, the Ricci tensor is parallel, provided $n > 2$, which together with (3.9) yields $r_k = 0 = r_{\bar{k}}$. Hence from (3.12) and (3.13), we obtain $R_{\bar{i}jk\bar{l}m} = 0$ and $R_{\bar{i}jk\bar{l}\bar{m}} = 0$, that is, M is locally symmetric. Conversely, if M is locally symmetric, then $S_{\bar{j}ik} = 0, S_{\bar{j}i\bar{k}} = 0, r_k = 0$ and $r_{\bar{k}} = 0$. Thus, from (3.12) and (3.13), the conformal curvature tensor field of M is parallel. Hence we have

THEOREM 3.5. *Let M be an indefinite Kaehlerian manifold of complex dimension $n (n > 2)$. Then M is locally symmetric if and only if the conformal curvature tensor field of M is parallel.*

Let M' be an $(n+p)$-dimensional indefinite Kaehlerian manifold of index $2(s+t)$ with vanishing conformal curvature tensor field and let M be an n-dimensional indefinite complex submanifold of index $2s$. Restricting the Riemannian curvature tensor R' of M' to M, we get

$$(3.14)\ R'_{\bar{i}jk\bar{l}} = \frac{1}{n+p}(\varepsilon_j\delta_{ij}S'_{\bar{l}k} + \varepsilon_k S'_{\bar{i}j}\delta_{kl}) - \frac{(n+p+2)r'}{2(n+p)^2(n+p+1)}\varepsilon_j\varepsilon_k\delta_{ij}\delta_{kl} + \frac{r'}{2(n+p)(n+p+1)}\varepsilon_j\varepsilon_k\delta_{ik}\delta_{jl},$$

from which together with (2.11) it follows that

$$(3.15)\ S_{\bar{i}j} = \frac{1}{n+p}\{nS'_{\bar{i}j} + \frac{1}{2}(r'' - \frac{n^2+np+n-p}{(n+p)(n+p+1)}r')\varepsilon_j\delta_{ij}\} - (h_{\bar{i}j})^2,$$

where $r'' = 2\sum_k \varepsilon_k S'_{\bar{k}k}$.

Moreover summing up the equation (3.14) for i and j, we obtain

$$r = \frac{2}{n+p}\{nr'' - \frac{n(n^2+np+n-p)}{2(n+p)(n+p+1)}r'\} - h_2.$$

Taking account of (2.3), (2.10), (3.4), (3.14), (3.15) and the above equation, it is clear that the conformal curvature tensor field B_0 of M has the form

$$
(3.16)\quad B_{0,\bar{i}jk\bar{l}} = -\sum_x \varepsilon_x h^x_{\bar{j}k}\overline{h}^x_{il} + \frac{1}{n}\{\varepsilon_j\delta_{ij}(h_{\bar{l}k})^2 + \varepsilon_k(h_{\bar{i}j})^2\delta_{kl}\}
- \frac{n+2}{n^2(n+1)}h_2\varepsilon_j\varepsilon_k\delta_{ij}\delta_{kl} + \frac{1}{n(n+1)}h_2\varepsilon_j\varepsilon_k\delta_{ik}\delta_{jl}.
$$

From now on, we shall prove the following theorem as a necessary and sufficient condition that the conformal curvature tensor field of an indefinite complex hypersurfaces vanishes identically.

THEOREM 3.6. *Let M' be an $(n+1)$-dimensional indefinite Kaehlerian manifold of index $2(s+t), t = 0$ or 1, and with vanishing conformal curvature tensor field, and let M be an indefinite complex hypersurface of index $2s$ of M' $(n > 2)$. Then the following assertions are equivalent*

(1) *M has the vanishing conformal curvature tensor field,*
(2) *M is totally geodesic.*

Proof. Assume that the conformal curvature tensor field B_0 of M vanishes identically. Under this situation (3.16) reduces to

$$
(3.17)\quad h_{jk}\overline{h}_{il} = \frac{1}{n}\{\varepsilon_j\delta_{ij}(h_{\bar{l}k})^2 + \varepsilon_k(h_{\bar{i}j})^2\delta_{kl}\}
- \frac{n+2}{n^2(n+1)}h_2\varepsilon_j\varepsilon_k\delta_{ij}\delta_{kl} + \frac{1}{n(n+1)}h_2\varepsilon_j\varepsilon_k\delta_{ik}\delta_{jl},
$$

because of $p = 1$. Summing up the equation (3.17) for i and k, we have

$$
(h_{\bar{k}j})^2 = \frac{h_2}{n}\varepsilon_j\delta_{jk},
$$

because of $n > 2$. Substituting the above equation into (3.17), we obtain

$$
(3.18)\qquad h_{jk}\overline{h}_{il} = \frac{h_2}{n(n+1)}\varepsilon_j\varepsilon_k(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}).
$$

Transvecting $\varepsilon_i\varepsilon_l h_{il}$ to (3.18) and summing up the result for i and l, we have $(n+2)(n-1)h_2h_{jk}/n(n+1) = 0$. This implies that $h_2 = 0$ because of $n > 2$. Since $h_2 = 0$ and (3.18), we obtain $h_{jk}\overline{h}_{il} = 0$ for all indices. Thus this equation means that M is totally geodesic. The converse is trivial.

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