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FIBRED RIEMANNIAN SPACES WITH K–CONTACT STRUCTURE

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1. Introduction

The formal divergence δT of a tensor field T, say (T^i_{jk}) is defined by $\nabla_i T^i_{jk}$ in local coordinates. A Riemannian manifold M is said to have harmonic curvature if the formal divergence δR of the curvature tensor R vanishes identically. By means of the second Bianchi identity, $\delta R = 0$ if and only if the Ricci tensor S satisfies the equation

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = 0 \tag{1.1}$$

for arbitrary vector fields X, Y and Z. In general, a Riemannian manifold with parallel Ricci tensor has harmonic curvature but the converse is not true. Essential examples have been given in [1].

The Weyl conformal curvature tensor C and 3-tensor D in an m-dimensional Riemannian manifold are defined by

$$\begin{aligned}(1.2)\ C(X,Y)Z = R(X,Y)Z &- \{S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)AX - g(X,Z)AY\}/(m-2) \\ &+ r\{g(Y,Z)X - g(X,Z)Y\}/(m-1)(m-2), \\ (1.3)\ D(X,Y)Z = &-\{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\}/(m-2) \\ &+ \{(Xr)g(Y,Z) - (Yr)g(X,Z)\}/2(m-1)(m-2),\end{aligned}$$

respectively, where we have put $g(AX,Y) = S(X,Y)$ and r is the scalar curvature. It is well known that $\delta C = (m-3)D$ and a necessary and sufficient condition for a Riemannian manifold to be conformally flat is $C = 0$ for $m > 3$ or $D = 0$ for $m = 3$. The conformal curvature tensor C vanishes identically for $m = 3$. If the space is conformally

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flat of dimension > 3, then $D = 0$ is a consequence of $C = 0$ because of $\delta C = (m-3)D$. It follows from (1.1) that a Riemannian manifold M with harmonic curvature has the constant scalar curvature r because $\delta S = \frac{1}{2}\nabla r$, and that $D = 0$ by means of (1.3). The converse is clear. Thus we have

LEMMA 1. *A Riemannian manifold M has the harmonic curvature if and only if the scalar curvature r is constant and the Weyl conformal curvature 3-tensor D vanishes identically.*

2. K-contact manifolds with harmonic curvature

We suppose that M is a K-contact manifold of dimension m, that is, a manifold with contact metric structure (ϕ, ξ, η, g) and ξ is a Killing vector with respect to g. It is characterized by the equations

(2.1)

$$\phi\xi = 0,\ \phi^2 X = -X + \eta(X)\xi,\ \eta(X) = g(\xi, X),\ \eta(\xi) = 1,\ \nabla_X \xi = \phi X.$$

Since ξ is Killing and (2.1), we get

$$(\nabla_X \phi)Y + R(\xi, X)Y = 0 \tag{2.2}$$

for arbitrary vector fields X and Y. The fundamental 2-form Φ of the contact metric structure is defined by $\Phi(X,Y) = g(\phi X, Y)$.

By use of the equations (2.1) and (2.2), the Riemannian curvature tensor R and the Ricci tensor S of M satisfy the equations

$$R(\xi, X)\xi = -X + \eta(X)\xi, \tag{2.3}$$

and

$$S(X, \xi) = (m-1)\eta(X). \tag{2.4}$$

We shall prove the following

THEOREM 2. *In a K-contact manifold M of dimension $m \geq 3$, the following conditions are equivalent to one another:*

(1) *M is an Einstein manifold,*

(2) *the Weyl conformal curvature 3-tensor D vanishes identically,*

(3) *the curvature is harmonic,*
(4) $R(X,Y)S = 0$.

Proof. The condition (1) implies trivially (2), (3) and (4), and Lemma 1 means that the condition (3) implies (2). Hence, it is sufficient to show that both the conditions (2) and (4) imply (1). If $D = 0$ on a K-contact manifold M, then we have
(2.5)

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{2(m-1)}\{(Xr)g(Y,Z) - (Yr)g(X,Z)\}.$$

Differentiating (2.4) covariantly, we get

$$(2.6) \qquad (m-1)\Phi(X,Y) = (\nabla_X S)(Y,\xi) + S(\phi X, Y).$$

If we take the skew symmetric part in X and Y, then we have

$$(2.7) \quad (m-1)\Phi(X,Y) = \frac{1}{4(m-1)}\{(Xr)\eta(Y) - (Yr)\eta(X)\} + \frac{1}{2}\{S(\phi X, Y) - S(\phi Y, X)\}$$

by use of (2.5).

Since the structure vector ξ is a Killing one on M, we see $\mathcal{L}_\xi r = \xi r = 0$. Replacing X with ξ in (2.7), we obtain $Yr = 0$, that is, the scalar curvature r is constant. Hence, by means of Lemma 1, M has a harmonic curvature and, by use of (2.5) and (2.6), we have

$$(2.8) \qquad (\nabla_\xi S)(X,Y) = (m-1)\Phi(X,Y) - S(\phi X, Y).$$

Since ξ is Killing on M, S satisfies the equation

$$(2.9) \qquad (\mathcal{L}_\xi S)(X,Y) = (\nabla_\xi S)(X,Y) + S(\phi X, Y) + S(X, \phi Y) = 0.$$

By means of (2.4), (2.8) and (2.9), we see that $S(X,Y) = (m-1)g(X,Y)$, that is, M is an Einstein manifold.

In the mean time, the condition (4) is equivalent to

$$(2.10) \qquad S(R(X,Y)Z, U) + S(Z, R(X,Y)U) = 0.$$

Replacing U with ξ in (2.10) and using (2.4), we get

$$(m-1)\eta(R(X,Y)Z)+S(Z,R(X,Y)\xi)=0. \tag{2.11}$$

Replacing again X with ξ in (2.11) and making use of (2.3) and (2.4), we obtain

$$\begin{aligned} 0 &= (m-1)g(R(\xi,Y)Z,\xi)+S(Z,-Y+\eta(Y)\xi) \\ &= (m-1)g(Y,Z)-S(Y,Z), \end{aligned}$$

that is, M is Einstein. Thus the proof is completed.

A conformally flat Einstein manifold of dimension $m \geq 4$ is of constant curvature, and so is a 3-dimensional Einstein manifold. As a consequence of Theorem 2, we have the following result due to S. Tanno [4]:

COROLLARY 3. *A conformally flat K-contact manifold of dimension $m \geq 3$ is of constant curvature.*

3. Main Theorems

Let $\{M,B,g,\pi\}$ be a fibred Riemannian space, that is, M an m-dimensional total space with projectable Riemannian metric g, B an n-dimensional base space, and $\pi : M \to B$ a projection with maximal rank n. The fibre passing through a point of M is a p-dimensional submanifold of M and we denote it by $\overline{M}$, where $p = m-n$. A vector field on M is said to be vertical or horizontal if it is always tangent or orthogonal to fibres, respectively. Throughout this section U,V,W,W' will always be vertical vector fields, X,Y,Z,Z' basic ones, and we shall write $<\ ,\ >$ for the inner product with respect to g.

Let ∇ be the Riemannian connection of the total space M, h the second fundamental form and L the normal connection of each fibre $\overline{M}$ in M. In comparison of our notations with those of B. O'Neill [3],

$$h_U V = T_U V,\ h_V X = -T_V X,\ L_X Y = -A_X Y,\ L_X V = A_X V.$$

If the curvature tensor R is defined by

$$R(E,F)G = \nabla_E \nabla_F G - \nabla_F \nabla_E G - \nabla_{[E,F]} G$$

for any vector fields E, F and G on M, then the structure equations of the fibred Riemannian space M are given by [2,3,5]

$$\begin{aligned}
(3.1)\qquad \langle R(X,Y)Z,Z'\rangle &= \langle \hat{R}(X,Y)Z,Z'\rangle + 2\langle L_X Y, L_Z Z'\rangle \\
&\quad + \langle L_X Z, L_Y Z'\rangle - \langle L_Y Z, L_X Z'\rangle,\\
(3.2)\qquad \langle R(X,Y)Z,U\rangle &= \langle (\nabla_Z L)_X Y, U\rangle - \langle L_X Y, h_U Z\rangle \\
&\quad - \langle L_X Z, h_U Y\rangle + \langle L_Y Z, h_U X\rangle,\\
(3.3)\qquad \langle R(X,U)Y,V\rangle &= -\langle (\nabla_X h)_U V, Y\rangle + \langle (\nabla_U L)_X Y, V\rangle \\
&\quad - \langle L_X U, L_Y V\rangle + \langle h_V Y, h_U X\rangle,\\
(3.4)\qquad \langle R(U,V)W,W'\rangle &= \langle \overline{R}(U,V)W,W'\rangle + \langle h_U W, h_V W'\rangle \\
&\quad - \langle h_V W, h_U W'\rangle,
\end{aligned}$$

where $\hat{R}$ and $\overline{R}$ are the curvature tensors of B and $\overline{M}$, respectively.

In a fibred Riemannian space M with almost contact metric structure (ϕ, ξ, η, g), assume that ϕ is projectable and each fibre $\overline{M}$ is ϕ-invariant and tangent to ξ [5]. In this case, the base space B is an almost Hermitian space, the covariant structure tensor of which will be denoted by J, and each fibre $\overline{M}$ is an almost contact metric space with structure $(\overline{\phi}, \overline{\xi}, \overline{\eta})$. Concerning such a fibred space, we proved in [5] the following

THEOREM 4. *A fibred almost contact metric space M with ϕ-invariant fibres tangent to ξ is K-contact if and only if the base space B is almost Kaehlerian, each fibre $\overline{M}$ is K-contact, and the structure tensor $\overline{\xi}$ satisfies $\nabla_X \overline{\xi} = 0$, $h_U \overline{\xi} = 0$ and $L = J \otimes \overline{\xi}$.*

Now we suppose that the fibred K-contact space M is conformally flat. By means of Corollary 3, M is of constant curvature, that is,

$$(3.5)\qquad R(E,F)G = c\{\langle F,G\rangle E - \langle E,G\rangle F\}$$

with a constant $c = r/m(m-1)$, r being the scalar curvature of M. By means of (3.1), (3.4), (3.5) and Theorem 4, we get

$$\begin{aligned}
(3.6)\quad \langle \hat{R}(X,Y)Z,Z'\rangle &= c\{\langle Y,Z\rangle\langle X,Z'\rangle - \langle X,Z\rangle\langle Y,Z'\rangle\}\\
&\quad - J(X,Z)J(Y,Z') + J(Y,Z)J(X,Z') - 2J(X,Y)J(Z,Z'),
\end{aligned}$$

$$(3.7)\quad \langle \overline{R}(U,V)W,W'\rangle = c\{\langle V,W\rangle\langle U,W'\rangle - \langle U,W\rangle\langle V,W'\rangle\} - \langle h_U W, h_V W'\rangle + \langle h_V W, h_U W'\rangle.$$

By means of $\nabla_X \overline{\xi} = 0$ and $L = J \otimes \overline{\xi}$, we have

$$(3.8)\qquad \nabla_X L = (\nabla_X J) \otimes \overline{\xi},$$

and $h_U \overline{\xi} = 0$ implies

$$(3.9)\qquad \langle L_X Y, h_U Z\rangle = \langle (J_X Y)\overline{\xi}, h_U Z\rangle = 0$$

for any X, Y, Z and U. Substituting (3.5) and (3.9) into (3.2), we obtain $\nabla_Z L = 0$ or $\nabla_Z J = 0$, hence the base space B is Kaehlerian.

On the other hand, by use of $\nabla_X \overline{\xi} = 0$ and $h_U \overline{\xi} = 0$, we get $(\nabla_X h)_U \overline{\xi} = 0$. Hence, replacing V with $\overline{\xi}$ in (3.3) and contracting with respect to X and Y, we have $c = 1$, by means of (3.5). Thus we have

THEOREM 5. *If a fibred K-contact space M is conformally flat, then the base space is a complex space form.*

Moreover, by means of (3.4), for a fibred Riemannian space with conformal fibres, $h_U V = \langle U, V\rangle N, N$ being the mean curvature vector of $\overline{M}$ in M, and we obtain the following lemma due to S. Ishihara and M. Konishi [2].

LEMMA 6. *If a fibred Riemannian space with conformal fibres is of constant curvature, then each fibre is of constant curvature.*

Combining Corollary 3, Theorem 5 and Lemma 6, we have

THEOREM 7. *If a conformally flat fibred K-contact space M has conformal fibres, then the base space is a complex space form and each fibre is of constant curvature.*

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