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ON LORENTZIAN WARPED SPACE–TIME $R \times_f H$

JONG-CHUL PARK, JIN-HWAN KIM AND JONG-CHUL KIM

1. Preliminaries

Suppose (B, b) and (H, h) are semi-Riemannian manifolds and let f be a positive smooth function on B. The *warped product* $M = B \times_f H$ is the product manifold $B \times H$ furnished with metric tensor $g = \pi^*(b) + (f \circ \pi)^2 \sigma^*(h)$ where π and σ are the projections of $B \times H$ onto B and H respectively. Explicitly, if x is tangent to $B \times H$ at (p, q), then $g(x, x) = b(d\pi(x), d\pi(x)) + f^2(p)h(d\sigma(x), d\sigma(x))$. A warped product $B \times_f H$ is a *Lorentzian warped product* if B is a Lorentzian manifold and H is a Riemannian manifold.

The set of the lifts of all vector fields on B and H to $B \times H$ are denoted as usual by $L(B)$ and $L(H)$ respectively. Typically we use the same notation for a vector field and for its lift. Many geometric properties of $M = B \times_f H$ can be expressed in terms of warping function f and the geometries the base B and the fiber H of M.

The standard space-time models of the universe are warped products ([3] Chapter 12), as are the simplest models of neighborhoods of stars and black holes ([3] Chapter 13).

In this paper, we shall consider Lorentzian warped products of the form $R \times_f H$, where (H, h) is a Riemannian $n(> 1)$-dimensional manifold and the metric tensor g is given by $g = -dt^2 \oplus f^2(t)h$. These warped products may be space-times. Throughout this paper, we assume that a space-time M is the Lorentzian warped product $R \times_f H$ and all of the terminologies and notations will be referred to O'Neill [3].

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2. Ricci properties of Lorentzian warped product $R \times_f H$

In this section, we study the Ricci flatness and Einsteinness of Lorentzian warped product $M = R \times_f H$ by investigating the warping function f.

Given $(t,p) \in M$, let $E_j \in T_p(H)$ for $1 \leq j \leq n$ such that $\{\partial/\partial t, E_1, \ldots, E_n\}$ forms a g-orthonormal basis for $T_{(t,p)}(M)$. The Hessian of f is given by $H^f = f''dt \oplus dt$ and the Laplacian of f is given by $\Delta f = -f''$.

LEMMA 2.1. *The Ricci tensor Ric of M is given by*

$$\mathrm{Ric}(X,Y) = -\frac{n}{f}H^f(X_1,Y_1) + \mathrm{Ric}_H(X_2,Y_2) + f(X_2,Y_2)\Big\{\frac{f''}{f} + (n-1)\frac{f'^2}{f^2}\Big\}$$

where $X_1, Y_1 \in L(R)$, $X_2, Y_2 \in L(H)$, $X = X_1 + X_2$, $Y = Y_1 + Y_2$, and Ric_H is denoted as the lift (pullback by π) of Ricci tensor of H.

Proof. By simple computation, it obtained from Corollary 7.43 of O'Neill [3].

THEOREM 2.2. *The Lorentzian warped product $M = R \times_f H$ is Ricci flat if and only if H is Ricci flat and the warping function f is a positive constant function.*

Proof. Assume that M is Ricci flat. Then by Lemma 2.1,

$$-\frac{n}{f}f'' = 0 \tag{1}$$

$$\frac{f''}{f} + (n-1)\frac{f'^2}{f^2} + \mathrm{Ric}_H(E_j,E_j) = 0 \tag{2}$$

Solving (1), $f(t) = C_1 t + C$, where C_1, C are arbitrarily constant. Since f is a positive function, f is a positive constant function. Substituting it into (2), $\mathrm{Ric}_H(E_j,E_j) = 0$ $(1 \leq j \leq n)$. Hence H is Ricci flat.

Conversely, if H is Ricci flat and f is a positive constant function, then $\mathrm{Ric}(\partial/\partial t, \partial/\partial t) = 0$ and $\mathrm{Ric}(E_j,E_j) = 0$ $(1 \leq j \leq n)$. Hence M is Ricci flat.

LEMMA 2.3. *$M = R \times_f H$ is Einstein if and only if H is Einstein and f satisfies*

(i) $f'' = \alpha f$ and

(ii) $n(n-1)(ff'' - f'^2) = S_H$

where S_H is the scalar curvature of H and α is a constant.

Proof. Suppose that M is Einstein with constant scalar curvature S. Then by Lemma 2.1,

$$\frac{nf''}{f} = -\frac{S}{n-1} \quad \text{and}$$
$$\frac{f''}{f} + (n-1)\frac{f'^2}{f^2} + \mathrm{Ric}_H(E_j, E_j) = \frac{S}{n-1}.$$

Thus

$$\frac{f''}{f} + (n-1)\frac{f'^2}{f^2} + \mathrm{Ric}_H(E_j, E_j) = \frac{nf''}{f}.$$

Hence

$$\mathrm{Ric}_H(E_j, E_j) = \frac{n-1}{f^2}(ff'' - f'^2) = (n-1)(ff'' - f'^2)h(E_j, E_j).$$

On the other hand, f''/f is constant and so $ff'' - f'^2$ is constant. Therefore H is an Einstein manifold with scalar curvature $S_H = n(n-1)(ff'' - f'^2)$. Also f satisfies $f'' = \alpha f$ where $\alpha = S'/\{n(n-1)\}$. Conversely, suppose that H is Einstein and f satisfies $f'' = \alpha f$ and $n(n-1)(ff'' - f'^2) = S_H$. Then

$$\mathrm{Ric}(\partial/\partial t, \partial/\partial t) = -nf''/f = -n\alpha = n\alpha g(\partial/\partial t, \partial/\partial t).$$

$$\begin{aligned}
\mathrm{Ric}(E_j, E_j) &= \frac{f''}{f} + (n-1)\frac{f'^2}{f^2} + \mathrm{Ric}_H(E_j, E_j) \\
&= \frac{f''}{f} + (n-1)\frac{f'^2}{f^2} + \frac{S_H}{n}h(E_j, E_j) \\
&= \frac{f''}{f} + (n-1)\frac{f'^2}{f^2} + \frac{(n-1)(ff'' - f'^2)}{f^2} \\
&= nf''/f = n\alpha = n\alpha g(E_j, E_j).
\end{aligned}$$

Hence M is Einstein.

LEMMA 2.4. *If $M = R \times H$ is Einstein, then* Ric $= \lambda g$ *for a constant $\lambda \geq 0$ and hence the scalar curvature S of M is $(n-1)\lambda \geq 0$.*

Proof. Suppose $\lambda < 0$. Since f satisfies $nf'' = \lambda f$, $f(t) = C_1 \cos\sqrt{k}t + C_2 \sin\sqrt{k}t$ where $k = -\lambda/n > 0$. This function is not a positive function. Thus there is no warping function making M Einstein.

THEOREM 2.5. *The Lorentzian warped product $M = R \times_f H$ is Einstein if and only if* (a) *H is Einstein, $S_H = 0$ and f is positive constant, or* (b) *H is Einstein, $S_H \geq 0$ and $f(t) = Ce^{kt} + [S_H/\{4n(n-1)k^2C\}]e^{-kt}$ for some constants $C(>0)$, $k(\neq 0)$.*

Proof. Suppose that M is Einstein, by Lemma 2.4, Ric $= \lambda g$ for a constant $\lambda \geq 0$. By Lemma 2.1, (i) in Lemma 2.3 is to be $f'' = \alpha f = \lambda f/n$.

If $\lambda = 0$, then solving (i) in Lemma 2.3, $f(t)$ is a positive constant function and by (ii), the scalar curvature S_H of H is zero.

Now if $\lambda > 0$, solving (i), $f(t) = C_1 e^{\sqrt{\alpha}t} + C_2 e^{-\sqrt{\alpha}t}$. Substituting it into (ii), we obtain $4C_1C_2 = S_H/\{n(n-1)\alpha\} \geq 0$. Since $f > 0$ for all $t \in R$, $C_1 + C_2 > 0$ and $C_1C_2 \geq 0$. Hence $S_H \geq 0$ and $f(t) = Ce^{kt} + [S_H/\{4n(n-1)k^2C\}]e^{-kt}$ where $C(>0)$ and $k(\neq 0)$ are constants. Conversely, the function f in the hypothesis satisfies (i) and (ii) in Lemma 2.3. Thus M is Einstein.

From Lemma 2.3 and Theorem 2.5, we have the following corollary.

COROLLARY 2.6. *Let $M = R \times_f H$ be an Einstein space-time. If the scalar curvature S_H of H is not zero, then $f(t)$ is not of the form Ce^{kt} where $C > 0$ and $k \in R$.*

3. Null geodesic completeness of $R \times_f H$

In this section, we study the null geodesic completeness of Lorentzian warped products $M = R \times_f H$ in case that it is Einstein.

A space-time N is said to be *null (resp., timelike, spacelike) geodesically complete* if all null (resp., timelike, spacelike) geodesics may be defined for all values $-\infty < t < \infty$ of an affine parameter.

A space-time N is said to be *future (resp., past) null geodesically complete* if it is all future-directed (resp., past-directed) null geodesics can be defined for arbitrary positive (resp., negative) values of an affine parameter. N is said to be *geodesically complete* if all geodesics can be defined on the entire real line R. Metric completeness and geodesic completeness are unrelated for arbitrary Lorentzian manifolds (cf. Hopf-Rinow theorem for Riemannian manifolds). Also timelike geodesic completeness, null geodesic completeness, spacelike geodesic completeness are in equivalent.

THEOREM 3.1. *If the warping function f of $M = R \times_f H$ is constant, then H is complete if and only if M is geodesically complete.*

Proof. By Proposition 7.38 of O'Neill [3], all geodesics of M are either (up to parameterization) of the form $(at, \beta(t))$, $(t_0, \beta(t))$ or (at, p_0) where $a, t_0 \in R$, $p_0 \in H$ and β is a unit speed geodesic in H.

LEMMA 3.2. *Let H be a complete Riemannian manifold. Then $M = R \times_f H$ is future (resp., past) null geodesically complete iff $\int_0^\infty f(t)dt$ (resp., $\int_{-\infty}^0 f(t)dt$) is infinite.*

Proof. From Theorem 2.57 and Remark 2.58 of Beem and Ehrlich [1], we have it.

THEOREM 3.3. *Let $M = R \times_f H$ be an Einstein space-time and H be a complete Riemannian manifold. Then*

(1) *M is future or past null geodesically complete.*

(2) *M is null geodesically complete iff f is not of the form Ce^{kt} where $C(>0)$ and $k(\neq 0)$ are constants.*

Proof. By Lemma 2.5, the warping function f is of the following forms,

(a) a positive constant function, (b) Ce^{kt} or (c) $C_1e^{kt} + C_2e^{-kt}$ where C, C_1 and C_2 are positive constants and k is a nonzero constant.

Now apply Lemma 3.2 to each case.

Cases (a), (c): M is null geodesically complete.

$k > 0$ in case (b): M is future null geodesically complete but past null geodesically incomplete.

$k < 0$ in case (b): M is past null geodesically complete but future null geodesically incomplete.

Using Theorem 3.2 and Corollary 2.6, we have the following.

COROLLARY 3.4. *Let $M = R \times_f H$ be an Einstein space-time with H complete Riemannian manifold. If the scalar curvature S_H is not zero, then M is null geodesically complete.*

References

1. Beem, J. K. and P. E. Ehrlich, *Global Lorentzian geometry*, Marcel Dekker, New York, 1981.
2. Bishop, R. L. and B. O'Neill, *Manifold of negative curvature*, Trans. Amer. Math. Soc., **145** (1969), 1–49.
3. O'Neill B., *Semi-Riemannian geometry*, Academic Press, New York, 1983.

Department of Mathematics
Yeungnam University
Kyungsan 713–749, Korea

Department of Mathematics
Semyung University
Chechon 390–230, Korea