Comm. Korean Math. Soc. 8 (1993), No. 3, pp. 373–382

BOREL'S THEOREM ON NORMAL NUMBERS MODULO 2

JOONG S. HONG AND GEON H. CHOE

1. Spectrum and uniform distribution

Let (X,μ) be a probability space. A measurable transformation $\tau : X \to X$ is said to be measure preserving or μ-invariant if $\mu(\tau^{-1}E) = \mu(E)$ for every measurable set E. A μ-invariant transformation τ on X is called ***ergodic*** if $\mu(\tau^{-1}E\Delta E) = 0$ implies that $\mu(E) = 0$ or 1. For example, irrational rotations on the unit circle are ergodic. If τ is ergodic and if $f(\tau x) = f(x)$ for almost every $x \in X$, then f is constant almost everywhere. A measure preserving transformation τ is said to be *mixing* if $\mu(\tau^{-n}E \cap F)$ converges to $\mu(E)\mu(F)$ as n tends to infinity. Mixing transformations are ergodic.

Consider the behavior of the sequence $\sum_{k=0}^{n-1} \chi_E(\tau^k x)$ which equals the number of times that the points $\tau^k x$ visit the set E. The Birkhoff Ergodic Theorem for ergodic transformations implies that the relative frequency of visits is proportional to the size of the set E, that is, $\frac{1}{n}\sum_{k=0}^{n-1} \chi_E(\tau^k x) \to \mu(E)$ almost everywhere.

One of the classical examples is the Kronecker-Weyl Theorem on uniform distribution of integral multiples of an irrational number modulo one. Another example is the Borel's Theorem on normal numbers which states that if $x \in [0,1)$ is expanded into a form $\sum_{n=1}^{\infty} a_n 2^{-n}$, $a_n = 0, 1$, then for almost every x, the number of 1's in the first n digits of binary expansion of x is approximately $\frac{1}{2}$ for sufficiently large n. To see this, consider the mixing transformation τ on $[0,1)$ defined by $\tau x = 2x$ modulo 1 and note that $a_n = 1$ if and only if $\tau^n x \in [1/2, 1)$. For the general results on uniform distribution, see [5].

In this paper, we are interested in the uniform distribution of the sequence $y_n \in \{0, 1\}$ defined by $y_n(x) \equiv \sum_{k=0}^{n-1} \chi_E(\tau^k x) \pmod 2$, at

Received June 15, 1993. Revised April 18, 1993.
The authors wish to thank the referee for his/her helpful suggestions

each point x of X for a measurable subset E in X when $\tau x = 2x$ modulo 1 on $[0, 1)$. We want to know if the limit exists and equals $\frac{1}{2}$. Contrary to our intuition, the limit does not necessarily exist and even when it exists it is not equal to $\frac{1}{2}$, in general. This type of problem was first studied by Veech[9]. He considered the case when the transformations are given by irrational rotations on the unit circle and obtained results which showed that the length of the interval E and the rotational angle θ are closely related. For example, he proved that when the angle θ has bounded partial quotients in its continued fraction expansion, the sequence y_n is evenly distributed between 0 and 1 if and only if the length of the interval is not an integral multiple of θ modulo 1. For related results, see [1], [2], [7], [8].

First, we consider general ergodic transformations on a probability space (X, μ). Now we define an isometry U on $L^2(X)$ by

$$(Uf)(x) = \exp(\pi i \chi_E(x)) f(\tau x), \quad f \in L^2(X).$$

Then for $n \geq 1$,

$$(U^n f)(x) = \exp(\pi i \sum_{k=0}^{n-1} \chi_E(\tau^k x)) f(\tau^n x)$$

and for the constant function 1,

$$(U^n 1)(x) = \exp(\pi i \sum_{k=0}^{n-1} \chi_E(\tau^k x)) = \exp(\pi i y_n(x)),$$

and now our problem is to study the existence of

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (U^n 1)(x).$$

By the von Neumann's Mean Ergodic Theorem, we see that $\frac{1}{N}\sum_{n=1}^{N}(U^n 1)(x)$ converges to $P_H 1$ in $L^2(X)$ where P_H is the orthogonal projection onto the invariant subspace $H = \{h \in L^2(X) : Uh = h\}$.

If an isometry $Uf(x) = A(x)f(\tau x)$, $|A(x)| = 1$ a.e., has an eigenvalue λ, then we can choose $q \in L^2(X)$ such that $||q||_2 = 1$, $Uq = \lambda q$. Hence

$A(x)q(x) = \lambda q(\tau x)$. Since $|A(x)| = 1$, we have $|q(x)| = |\lambda|\,|q(\tau x)|$, and $|q|$ is an eigenfunction of U and $|\lambda|$ is an eigenvalue. Since τ is ergodic, we see that $|q|$ is constant a.e. and $|\lambda| = 1$. Hence A is of the form $A(x) = \lambda\overline{q(x)}q(\tau x)$. Recall that a function $f(x)$ is called a *coboundary* if $f(x) = \overline{q(x)}q(\tau x)$ where $|q(x)| = 1$ a.e. on X. Therefore U has an eigenvalue if $A(x)$ is a constant multiple of a coboundary. For an invertible ergodic transformation τ we have the induced unitary operator U_τ defined by $U_\tau f(x) = f(\tau x)$. In this case, $(Uf)(x) = A(x)f(\tau x)$, $|A(x)| = 1$ a.e., is also unitary. For further details in the case that the transformation is given by an irrational rotation, see [3], [4]. And for recent results which extend [2], see [6].

Let τ be an ergodic transformation on X and let U be the isometry on $L^2(X)$ given by $Uf(x) = A(x)f(\tau x)$ where $A(x)$ is real-valued and $|A(x)| = 1$ a.e. on X. Define a subspace $H = \{h \in L^2(X) : Uh = h\}$. Then the dimension of H is 0 or 1. If $\dim H = 0$, then

$$\frac{1}{N}\sum_{n=1}^{N} U^n 1 \to 0 \text{ in } L^2(X).$$

If $\dim H = 1$, then $A(x)$ is a coboundary. And its converse is also true. In this case, if we take $q \in H$, $||q||_2 = 1$, then $|q(x)| \equiv 1$ a.e. and $A(x) = \overline{q(x)}q(\tau x)$. Furthermore, we may choose a real valued function for such q. In this case,

$$\frac{1}{N}\sum_{n=1}^{N} U^n 1 \to \int_X q(x)\,d\mu \cdot q \quad \text{in } L^2(X).$$

2. Borel's theorem modulo 2

Now we consider Borel's theorem modulo 2 for the case $E = [1/2, 1)$. We identify the half-open interval $[0,1)$ with the unit circle $|z| = 1$. Hence $[a,b) = [a,1) \cup [0,b)$ for $a > b$. Define $Uf(x) = \exp(\pi i \chi_{[1/2,1)}(x))$ $f(\tau x)$ where $\tau : [0,1) \to [0,1)$ is defined by $\tau x = 2x \pmod 2$. Let $H = \{h \in L^2(X) : Uh = h\}$. Then we can easily show $\dim H = 0$. Hence we conclude that $\frac{1}{N}\sum_1^N U^n 1$ converges to 0 in $L^2(\mathbb{T})$.

Now we consider τ on $[0,1)$ with invariant measure μ_p and $Uf(x) = \exp(\pi i \chi_{[1/2,1)})f(\tau x)$ in $L^2(X,\mu_p)$. Since the one-sided shift is mixing for any p, we see that the invariant space H is $\{0\}$. Therefore we may conclude that modulo 2 theorem on normal numbers is true with respect to any measure μ_p.

Note that for the set $[\frac{1}{6},\frac{5}{6}]$, the function $q = \exp(\pi i \chi_I)$ is a coboundary since $I = E \triangle \tau^{-1}E$ for $E = [\frac{1}{3},\frac{2}{3}]$. Hence in this case we have irregularities in the distribution of y_n since $\int q\,dx \neq 0$. And note that for $I = [\frac{1}{4},\frac{3}{4})$, $q = \exp(\pi i \chi_I)$ is a coboundary since $I = E \triangle \tau^{-1}E$ for $E = [\frac{1}{2},1)$. But we have $\int q\,dx = 0$, hence we obtain the uniform distribution modulo 2 in this case even though the invariant subspace is not trivial.

3. The Main Result

From now on, the numbers $j,k,m,n,s,s_0,t,m,n,N,N_1,N_1^*,N_\sigma,M,$ $M_0,M_1,M_\sigma,L_1,L_\sigma,K_0,K_1$ are positive integers. And by the abuse of notations the relation $A = B$ denotes $A = B$ modulo measure zero sets and $A \subset B$ denotes $A \subset B$ modulo measure zero sets for any measurable sets A, B.

For $s > 0$, let $\{0,1\}^{(s)}$ be the set of all real numbers of the form $\sum_{n=1}^{s} a_n 2^{-n}$, $a_n = 0,1$. Denote $\sum_{n=1}^{s} a_n$ by $(a_1 a_2 \cdots a_s)$.

Recall that for a measurable subset of the real line E, the Lebesgue density theorem states that the *metric density* $d_E(x)$ of E at x defined by

$$\lim_{r\to 0+} \frac{\mu(E\cap(x-r,x+r))}{\mu(x-r,x+r)}$$

is equal to 1 for a.e. $x \in E$ and equal to 0 for a.e. $x \notin E$. We may define the *right* metric density using the interval $(x,x+r)$.

THEOREM. *If two real numbers a and b satisfy $a < b$, $a,b \in \{0,1\}^{(s)}$ for some $s > 0$ and $I = [a,b] \subset [\frac{1}{2},1)$, then $\exp(\pi i \chi_I)$ is not a coboundary, hence we have uniform distribution modulo 2.*

Proof. Suppose $\exp(\pi i \chi_I)$ is a coboundary, then there exists a measurable set E such that $I = [a,b] = E \triangle \tau^{-1}E$ modulo measure zero set. Define mappings f_0, f_1 on $[0,1)$ by $f_0(x) = x/2$, $f_1(x) = x/2 + 1/2$ for $x \in [0,1)$. Then $\tau^{-1}(x)$ consists of two points $f_0(x)$ and $f_1(x)$, hence

$\tau^{-1}E = f_0E \cup f_1E$ which is a disjoint union. Note that I can be decomposed into a disjoint union $I = E \,\Delta\, \tau^{-1}E = (f_0E - E) \cup \{E - (f_0E \cup f_1E)\} \cup (f_1E - E)$.

Since $E - f_0E = ([0, \frac{1}{2}] \cap (E - f_0E)) \cup ([\frac{1}{2}, 1) \cap E)$ $(\because f_0E \subset [0, \frac{1}{2}])$ and $E - f_1E = \{[0, \frac{1}{2}] \cap E\} \cup \{[\frac{1}{2}, 1) \cap (E - f_1E)\}$, so

$$\begin{aligned} E - (f_0E \cup f_1E) &= (E - f_0E) \cap (E - f_1E) \\ &= \{[0, \frac{1}{2}] \cap (E - f_0E)\} \cup \{[\frac{1}{2}, 1) \cap (E - f_1E)\}. \end{aligned} \tag{1}$$

And $E-(f_0E \cup f_1E) \subset I \subset [\frac{1}{2}, 1]$ implies $\mu([0, \frac{1}{2}] \cap (E - f_0E)) = 0$. And $f_0E - E \subset f_0E \subset [0, \frac{1}{2}], f_0E - E \subset I \subset [\frac{1}{2}, 1]$ implies $\mu(f_0E - E) = 0$, which in turn implies $f_0E \subset E$. Since $f_1 : [0, 1) \to [\frac{1}{2}, 1)$ is a bijection, $[\frac{1}{2}, 1) \cap (f_1E)^c = f_1(E^c)$. So the interval $I = [a, b]$ can be written as

$$I = (f_1E - E) \cup \{[\frac{1}{2}, 1) \cap (E \cap (f_1E)^c)\} = (E^c \cap f_1E) \cup (E \cap f_1(E^c)). \tag{2}$$

Now the remainder of the proof is split up into six steps.

Step 1. Both $f_1E - E$ and $[\frac{1}{2}, 1) \cap (E - f_1E) = E \cap f_1(E^c)$ have positive measure.

Proof. (i) Suppose $\mu(f_1E - E) = 0$, then $I = [a, b] = [\frac{1}{2}, 1) \cap (E - f_1E) = E \cap f_1(E^c)$, so $[a, b] \subset f_1(E^c)$ and $f_1^{-1}[a, b] = [2a-1, 2b-1) \subset E^c$.

Let $f_\sigma E = f_{c_1} f_{c_2} \cdots f_{c_n} E$ for $\sigma = (c_1\, c_2 \cdots c_n) \in \{0, 1\}^{(n)}$. Then $f_0E \subset E$ and $f_1E \subset E$ implies $f_\sigma E \subset E$ for any $\sigma \in \{0, 1\}^{(n)}$ and for any n. It implies that we can find n and $\sigma \in \{0, 1\}^{(n)}$ such that $\mu(f_\sigma E \cap [2a - 1, 2b - 1)) > 0$, which is a contradiction.

(ii) Suppose $\mu\{[\frac{1}{2}, 1) \cap (E - f_1E)\} = 0$, then $\mu\{E - (f_0E \cup f_1E)\} = 0$ and $E \subset f_0E \cup f_1E = \tau^{-1}E$. Since τ is ergodic, $\mu(E) = (\tau^{-1}E)$. It implies $E = \tau E$, so $\mu(E)$ is 0 or 1, which is a contradiction.

Now let $f_1E - E = A$, $[\frac{1}{2}, 1) \cap (E - f_1E) = B$, and $\mu(A) = \alpha, \mu(B) = \beta$ where $\alpha > 0, \beta > 0, \alpha + \beta = \mu(I) = b - a$. Without loss of generality, we may assume $\alpha \geq \beta$. Define $f_0^n x$ as n-th iterate of f_0 at x.

Step 2. For all $n > 0$, $f_0^n A \subset E^c$, $f_0^n B \subset E$ where $f_0^n I = f_0^n A \cup f_0^n B$.

Proof. i) If $A \subset E^c$, then $f_0A \subset f_0(E^c) = (f_0E)^c \cap [0, \frac{1}{2}]$ and $f_0A \cap E \subset (E - f_0E) \cap [0, \frac{1}{2}]$. Since $\mu\{[0, \frac{1}{2}] \cap (E - f_0E)\} = 0$, we have $\mu(f_0A \cap$

$E) = 0$, and $f_0 A \subset E^c$. Continuing the same method, we conclude that $f_0^n A \subset E^c$ for all n.

ii) If $B \subset E$, then $f_0 B \subset f_0 E$ and $f_0 B \cap E^c \subset f_0 E - E$. Since $\mu(f_0 E - E) = 0$, we have $\mu(f_0 B \cap E^c) = 0$ and $f_0 B \subset E$. Similarly, $f_0^n B \subset E$ for all n.

Step 3. For a measurable subset $D \subset [0,1)$ the following hold:

i) If $D \subset E$, then $f_0 D \subset E$. If $D \subset E^c$, then $f_0 D \subset E^c$.

ii) If $f_1 D \subset I$, and $D \subset E$, then $f_1 D \subset E^c$. If $f_1 D \subset I$, and $D \subset E^c$, then $f_1 D \subset E$.

iii) If $f_1 D \subset I^c$, and $D \subset E$, then $f_1 D \subset E$. If $f_1 D \subset I^c$, and $D \subset E^c$, then $f_1 D \subset E^c$.

Proof. i) If $D \subset E$, then $f_0 D \subset f_0 E \subset E$. If $D \subset E^c$, then $f_0 D \subset f_0(E^c)$. So $f_0 D \cap E \subset E \cap f_0(E^c)$. Since $E \cap f_0(E^c)$ has measure zero by Step 1, we have $\mu(f_0 D \cap E) = 0$ and $f_0 D \subset E^c$.

ii) If $f_1 D \subset I$ and $D \subset E$, then $f_1 D \subset f_1 E \cap I = A = f_1 E - E$. If $f_1 D \subset I$, and $D \subset E^c$, then $f_1 D \subset f_1(E^c) \cap I = B$.

iii) If $f_1 D \subset I^c$, and $D \subset E$, then $f_1 D \subset f_1 E \cap I^c = f_1 E \cap A^c = f_1 E \cap E$. If $f_1 D \subset I^c$ and $D \subset E^c$, then

$$\begin{aligned} f_1 D \subset f_1(E^c) \cap I^c &\subset f_1(E^c) \cap B^c \\ &= \{(f_1 E)^c \cap [\tfrac{1}{2}, 1)\} \cap \{E^c \cup f_1 E \cup [0, \tfrac{1}{2}]\} \\ &= (f_1 E)^c \cap [\tfrac{1}{2}, 1) \cap E^c. \end{aligned}$$

Step 4. If $0 < \epsilon \leq b$, then

$$\frac{\beta}{b} \leq \frac{\mu([0,\epsilon] \cap E)}{\epsilon} \leq 1 - \frac{\alpha}{b}.$$

Proof. Since $b/2 \leq a$, all $f_1^n I$'s are mutually disjoint. If $0 < \epsilon \leq b$, then for such ϵ, there exists M_0 such that $b/2^{M_0} < \epsilon \leq b/2^{M_0 - 1}$. So $f_0^n I \subset [0,\epsilon]$ for $n \geq M_0$, and by Step 2, $f_0^n A \subset E^c, f_0^n B \subset E$ for $n \geq M_0$.

Hence

$$\mu({f_0}^{M_0} B \cup {f_0}^{M_0+1} B \cup \cdots) \leq \mu([0,\epsilon] \cap E) \leq \epsilon - \mu({f_0}^{M_0} A \cup {f_0}^{M_0+1} A \cup \cdots),$$

$$\frac{\beta}{b} \le \frac{\beta}{2^{M_0-1}\epsilon} \le \frac{\mu([0,\epsilon]\cap E)}{\epsilon} \le 1 - \frac{\alpha}{2^{M_0-1}\epsilon} \le 1 - \frac{\alpha}{b}.$$

Step 5. If $a > \frac{1}{2}$, then there exists integer N_1 such that $f_1 f_0^n A \subset E^c$ and $f_1 f_0^n B \subset E$ for $n \ge N_1$. And if $0 < \epsilon \le b/2^{N_1+1}$, then

$$\frac{\beta}{b} \le \frac{\mu([\frac{1}{2}, \frac{1}{2}+\epsilon]\cap E)}{\epsilon} \le 1 - \frac{\alpha}{b}.$$

If $a = \frac{1}{2}$, then there exists integer N_1^* such that $f_1 f_0^n A \subset E$ and $f_1 f_0^n B \subset E^c$ for $n \ge N_1^*$. And if $0 < \epsilon \le b/2^{N_1^*+1}$, then

$$\frac{\alpha}{b} \le \frac{\mu([\frac{1}{2}, \frac{1}{2}+\epsilon]\cap E])}{\epsilon} \le 1 - \frac{\beta}{b}.$$

Proof. If $a > \frac{1}{2}$, then there exists N_1 such that

$$f_1 f_0^n I \subset I^c \quad \text{for } n \ge N_1.$$

By Step 2 and Step 3 (iii), $f_1 f_0^n A \subset E^c$, $f_1 f_0^n B \subset E$ for $n \ge N_1$. If $0 < \epsilon \le b/2^{N_1+1}$, then there exists M_1 such that

$$\frac{b}{2^{M_1+1}} \le \epsilon \le \frac{b}{2^{M_1}}.$$

Since $f_1 f_0^n I = [\frac{1}{2} + \frac{a}{2^{n+1}}, \frac{1}{2} + \frac{b}{2^{n+1}}]$, so $f_1 f_0^n I \subset [\frac{1}{2}, \frac{1}{2}+\epsilon]$ for $n \ge M_1$. Hence,

$$\begin{aligned}\mu(f_1 f_0^{M_1} B \cup f_1 f_0^{M_1+1} B \cup \cdots) &\le \mu([1/2, 1/2+\epsilon]\cap E) \\ &\le \epsilon - \mu(f_1 f_0^{M_1} A \cup f_1 f_0^{M_1+1} A \cup \cdots),\end{aligned}$$

so

$$\frac{\beta}{b} \le \frac{\beta}{2^{M_1}\epsilon} \le \frac{\mu([\frac{1}{2}, \frac{1}{2}+\epsilon]\cap E)}{\epsilon} \le 1 - \frac{\alpha}{2^{M_1}\epsilon} \le 1 - \frac{\alpha}{b}.$$

The case of $a = \frac{1}{2}$ is similar to the above.

Step 6. For any m and $\sigma \in \{0,1\}^{(m)}$, there exist N_σ such that if $0 < \epsilon \leq b/2^{m+N_\sigma+1}$ then

$$\frac{\beta}{b} \leq \frac{\mu([f_\sigma(\frac{1}{2}), f_\sigma(\frac{1}{2})+\epsilon] \cap E)}{\epsilon} \leq 1 - \frac{\beta}{b}.$$

Proof. Given m and $\sigma = (c_1, \cdots, c_m) \in \{0,1\}^{(m)}$ with $c_j \in \{0,1\}$, then $f_\sigma(\frac{1}{2}) \in [a,b)$ or $f_\sigma(\frac{1}{2}) \in [0,a) \cup [b,1)$. So there exists N_σ such that

$$\begin{aligned} &f_\sigma f_1 f_0^n I \subset I \quad \text{for} \quad n \geq N_\sigma \quad \text{or} \\ &f_\sigma f_1 f_0^n I \subset I^c \quad \text{for} \quad n \geq N_\sigma. \end{aligned}$$

By Step 5 and Step 3,

$$\begin{aligned} &f_\sigma f_1 f_0^n A \subset E \quad \text{and} \quad f_\sigma f_1 f_0^n B \subset E^c \quad \text{for} \quad n \geq N_\sigma \quad \text{or} \\ &f_\sigma f_1 f_0^n A \subset E^c \quad \text{and} \quad f_\sigma f_1 f_0^n B \subset E \quad \text{for} \quad n \geq N_\sigma. \end{aligned}$$

By Step 3, there are six cases. But it suffices to prove only one of them because the remaining cases are identical. We consider the following case.

(1) $c_1 = 1$,
(2) $f_{c_2} f_{c_3} \cdots f_{c_n} f_1 f_0^n A \subset E$ and $f_{c_2} f_{c_3} \cdots f_{c_n} f_1 f_0^n B \subset E^c$ for $n \geq N_\sigma$,
(3) $f_\sigma f_1 f_0^n I \subset I$ for $n \geq N_\sigma$.

Then by Step 3 (ii), $f_\sigma f_1 f_0^n A \subset E^c$ and $f_\sigma f_1 f_0^n B \subset E$ for $n \geq N_\sigma$. Since $f_1 f_0^n I = [1/2 + a/2^{n+1}, 1/2 + b/2^{n+1}]$ and

$$f_1(x+y) = f_1 x + \frac{y}{2}, \quad f_0(x+y) = f_0 x + \frac{y}{2},$$

we have

$$f_\sigma f_1 f_0^n I = \left[f_\sigma(\frac{1}{2}) + \frac{a}{2^{n+m+1}}, f_\sigma(\frac{1}{2}) + \frac{b}{2^{n+m+1}} \right].$$

Let $0 < \epsilon \leq b/2^{m+N_\sigma+1}$. For such ϵ, there exists M_σ such that $b/2^{m+M_\sigma+1} \leq \epsilon \leq b/2^{m+M_\sigma}$. Then $f_\sigma f_1 f_0^n I \subset [f_\sigma(\frac{1}{2}), f_\sigma(\frac{1}{2}) + \epsilon]$ for $n \geq M_\sigma$. Hence

$$\begin{aligned} \mu(f_\sigma f_1 f_0^{M_\sigma} B \cup f_\sigma f_1 f_0^{M_\sigma+1} B \cup \cdots) &\leq \mu\left([f_\sigma(\frac{1}{2}), f_\sigma(\frac{1}{2}) + \epsilon] \cap E \right) \\ &\leq \epsilon - \mu(f_\sigma f_1 f_0^{M_\sigma} A \cup f_\sigma f_1 f_0^{M_\sigma+1} A \cup \cdots) \end{aligned}$$

and

$$\frac{\beta}{b} \le \frac{\beta}{2^{m+M_\sigma}\epsilon} \le \frac{\mu([f_\sigma(\frac{1}{2}), f_\sigma(\frac{1}{2})+\epsilon] \cap E)}{\epsilon}$$
$$\le 1 - \frac{\alpha}{2^{m+M_\sigma}\epsilon} \le 1 - \frac{\alpha}{b} \le 1 - \frac{\beta}{b}.$$

Step 7. Now as a final step we show that there is no measurable subset E for which $I = E \Delta \tau^{-1} E$. Without loss of generality, we may assume $a \in \{0,1\}^{(s)} - \{0,1\}^{(s-1)}$. For any $\sigma \in \{0,1\}^{(s)}$, there exists N_σ by Step 6. We can choose M such that $M \ge \max_{\sigma \in \{0,1\}^{(s)}} N_\sigma$, and $f_\sigma f_1 {f_0}^M(b) - f_\sigma(\frac{1}{2}) < 1/2^s$. Since $a \in \{0,1\}^{(s)} - \{0,1\}^{(s-1)}$ with $s \ge t$, so for any $\sigma \in \{0,1\}^{(m)}$ with $m \ge s$,

$$f_\sigma f_1 {f_0}^n I \subset I \quad \text{or} \quad f_\sigma f_1 {f_0}^n I \subset I^c \quad \text{for} \quad n \ge M. \tag{3}$$

Suppose the metric density $d_E(x)$ of E at x is 1 for $x \in [0,1)$, then the right metric density of E^c at x is 0. So for any $\delta > 0$, there exists $0 < \epsilon < (\frac{1}{2})^s$ such that $\mu([x, x+\epsilon] \cap E^c)/\epsilon \le \delta$. For such ϵ, there exists $s_0 \ge s$ such that $(\frac{1}{2})^{s_0+1} \le \epsilon < (\frac{1}{2})^{s_0}$. Then there exist $\sigma_1, \sigma_2 \in \{0,1\}^{(s_0+1)}$ such that $[f_{\sigma_1}(\frac{1}{2}), f_{\sigma_2}(\frac{1}{2})] \subset [x, x+\epsilon]$. Since $s_0 + 1 > s$, by (3)

$$f_{\sigma_1} f_1 f_0^n A \subset E \quad \text{and} \quad f_{\sigma_1} f_1 f_0^n B \subset E^c \quad \text{for} \quad n \ge M \quad \text{or} \tag{4}$$
$$f_{\sigma_1} f_1 f_0^n A \subset E^c \quad \text{and} \quad f_{\sigma_1} f_1 f_0^n B \subset E \quad \text{for} \quad n \ge M. \tag{5}$$

Suppose that (4) holds. Then we have

$$\begin{aligned} \mu([x, x+\epsilon] \cap E^c) &\ge \mu([f_{\sigma_1}(\frac{1}{2}), f_{\sigma_2}(\frac{1}{2})] \cap E^c) \\ &\ge \mu(f_{\sigma_1} f_1 f_0^M B \cup f_{\sigma_1} f_1 f_0^{M+1} B \cup \cdots) \\ &= \frac{\beta}{2^{s_0+M+1}}, \end{aligned}$$

so

$$\frac{\mu([x, x+\epsilon] \cap E^c)}{\epsilon} \ge \frac{\beta}{2^{s_0+M+1}\epsilon} \ge \frac{\beta}{2^{M+1}}.$$

Since $\delta > \beta/2^{M+1}$ and since δ can be chosen arbitrarily small, it is a contradiction. Now suppose that (5) holds true, then $\mu(E) = 0$, which is also a contradiction.

References

1. G. H. Choe, *Ergodicity and irrational rotations*, Proc. Royal Irish Acad. (to appear).
2. ———, *Products of operators with singular continuous spectra*, Proc. Sympos. in Pure Math. vol. 51, 65–68, Amer. Math. Soc., Providence, R.I. 1990.
3. ———, *Spectral Types of Multiplicative Cocycles*, Ph.D. Thesis, Univ. of California, Berkeley, 1987.
4. H. Helson, *Cocycles on the circle*, J. Operator Theory **16** (1986), 189–199.
5. L. Kuipers and H. Niederreiter, *Uniform Distributions of Sequences*, John Wiley and Sons, New York, 1974.
6. H. Medina, *Hilbert Space Operators Arising from Irrational Rotations on the Circle Group*, Ph.D. Thesis, Univ. of California, Berkeley, 1992.
7. K.D. Merrill, *Cohomology of step functions under irrational rotations*, Israel J. Math. **52** (1985), 320–340.
8. M. Stewart, *Irregularities of uniform distribution*, Acta Math. Sci. Hungar. **37** (1981), 185–221.
9. W.A. Veech, *Strict ergodicity in zero-dimensional dynamical systems and Kronecker-Weyl theorem mod 2*, Trans. Amer. Math. Soc. **140** (1968), 1–33.

Department of Mathematics
Korea Advanced Institute of Science and Technology
373-1 Kusongdong, Yusonggu
Taejon 305-701, Korea