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ON REDUCED RINGS AND GENERALIZED DUO RINGS

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1. Introduction

Throughout R will represent a ring with 1, and left or right R-modules are unitary. A ring R is called an ELT[9] if every essential left ideal is two-sided. Following [5], R is called a left SF-ring if every simple left R-module is flat. A left R-module M is called p-injective if for any principal left ideal I of R and left R-homomorphism $f : I \to M$, there exists $y \in M$ such that $f(x) = xy$ for all $x \in I$. Throughout this paper, we shall write "R satisfies $(*)$" if every left annihilator is two-sided ideal and "R satisfies $(**)$" if for left ideals I and J, $I \cap J = 0$ implies $IJ = 0$. Obviously every left duo ring (a ring in which every left ideal is two-sided ideal) satisfies $(*)$ and $(**)$. Every right duo ring and reduced ring satisfy $(*)$. In this article some rings satisfying $(*)$ and $(**)$ are studied. If $A \subseteq R$, the left annihilator of A will be denoted by $\ell(A)$ and the right annihilator of A by $r(A)$. The left singular ideal will be denoted by $Z_\ell(R)$.

2. Rings satisfying $(*)$

A ring R is called abelian if every idempotent is central. In this section we concerned with rings satisfying the condition $(*)$ every left annihilator is two-sided. Recall that every left (right) duo rings and reduced rings satisfy $(*)$.

We begin with a proposition.

PROPOSITION 2.1. *If R satisfies $(*)$ then R is an abelian ring.*

Proof. Let $e \in R$ be any idempotent. Since $Re = \ell(1-e)$ and $R(1-e) = \ell(e)$, it follows that Re and $R(1-e)$ are two-sided ideals in R.

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Then $eR \subseteq ReR \subseteq Re$ and so $eR(1-e) = 0$. Similarly $(1-e)R \subseteq R(1-e)R \subseteq R(1-e)$, hence $(1-e)Re = 0$. Given any $r \in R$, we thus have $er(1-e) = 0$ as well as $(1-e)re = 0$, whence $er = ere = re$. Therefore e is central.

COROLLARY 2.2. *Any left (right) duo rings and reduced rings are abelian.*

A ring R is called fully left idempotent if every left ideal is idempotent. Recall that a ring R is an ELT if every essential left ideal of R is two-sided. We also recall a result due to Ming[8] : every ELT fully right idempotent ring is (von Neumann) regular. But we don't know whether ELT full left idempotent ring is regular. However we have the following proposition which also extends [5, Proposition 3.2].

PROPOSITION 2.3. *Let R be a ring satisfying $(*)$. Then the following are equivalent :*

(a) *R is a strongly regular ring.*
(b) *R is a right SF-ring.*
(c) *R is a an ELT fully left idempotent ring.*

Proof. The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) are well-known. (b) $\Rightarrow$ (a): Proposition 2.1 and [5, Proposition 3.2]. (c) $\Rightarrow$ (a): We will prove that $Ra + \ell(a) = R$ for every $a \in R$. Suppose this is not true. Then there exists a maximal left ideal M containing $Ra + \ell(a)$ for some $a \in R$. First observe that M must be essential in R. For, if M is not essential, then $M = Re$ for some $e = e^2 \in R$. Now $a(1-e) = 0$; hence $(1-e)a = 0$ since R is abelian. So $1 - e \in \ell(a) \subseteq M$ which implies $1 \in M$, which is a contradiction. Therefore M is a two-sided ideal by assumption. Since R is a fully left idempotent, R/M is a flat right R-module. Thus $a = ba$ for some $b \in M$ by [1, Proposition 2.1]. Hence $1 - b \in \ell(a) \subseteq M$ which implies $1 \in M$, which is a contradiction. Therefore $Ra + \ell(a) = R$ for every $a \in R$ and $ra + s = 1$ for some $r \in R$ and $s \in \ell(a)$. This implies $a = ra^2$; hence R is strongly regular.

We add an elementary characterization of reduced ring.

PROPOSITION 2.4. *The following are equivalent :*

(a) *R is a reduced ring.*
(b) *R is a semiprime ring satisfying the condition $(*)$.*

(c) *R is a left nonsingular ring satisfying the condition $\ell(a) = r(a)$ for every $a \in R$.*

Proof. The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) are well-known. (b) $\Rightarrow$ (a) : Suppose $a^2 = 0$ for $a \in R$. Then $a \in \ell(a)$, so $aR \subseteq \ell(a)$ by the condition $(*)$. Thus $aRa = 0$, hence $(aR)^2 = 0$ and since R is semiprime, $aR = 0$. Therefore $a = 0$ as required. (c) $\Rightarrow$ (a) : Suppose that $a^2 = 0$ for $a \in R$. Then there exists a left ideal L such that $r(a) \oplus L$ is an essential left ideal in R. Since $aL \subseteq L \cap r(a) = 0$, $L \subseteq r(a) = \ell(a)$. Thus $r(a)$ is an essential left ideal which implies $a \in Z_\ell(R) = 0$.

We provide the following two examples, which are non-reduced rings satisfying the condition $(*)$.

EXAMPLE 1. Let Z be the ring of integers. Set $R = Z/4Z$. Obviously R satisfies the condition $(*)$. But R is not semiprime, since its only proper ideal is nilpotent.

EXAMPLE 2. Let F be a field and $F(x)$ the field of rational functions over F. Let $R = F(x) \times F(x)$ as an additive group and define the multiplication as follows :

$$(f_1(x), g_1(x))(f_2(x), g_2(x)) = (f_1(x)f_2(x), f_1(x^2)g_2(x) + g_1(x)f_2(x)).$$

It is trivial to show that R is a ring with 1. Moreover one can observe that $Ra \subseteq aR$ for any element $a \in R$, i.e., every principal right ideal of R is two-sided ideal. So R is a right duo ring. Therefore R satisfies the condition $(*)$. Now consider the ideal $(0, F(x))$ of R. Then $(0, F(x))^2 = 0$, so R is not semiprime.

3. Rings satisfying $(**)$

In this section we study rings with the condition $(**)$ for left ideals I and J of R, $I \cap J = 0$ implies $IJ = 0$. Obviously left duo rings satisfy $(**)$. Recall that R is called SLB (strongly left bounded) ring if every nonzero left ideal of R contains nonzero two-sided ideal.

PROPOSITION 3.1. *If R is a semiprime SLB ring, then R satisfies $(**)$.*

Proof. (suggested by G. F. Birkenmeier) Suppose that R does not satisfy $(**)$. Then there exists some left ideals I and J such that $I \cap J = 0$

and $IJ \neq 0$. Since R is a SLB ring, IJ contains nonzero two-sided ideal K. So $K \subseteq IJ \subseteq J$, hence $KI \subseteq I \cap K \subseteq I \cap J = 0$. Therefore $K^2 \subseteq K(IJ) = (KI)J = 0$. Since R is a semiprime ring, $K = 0$, a contradiction.

The following example shows that there exists a ring satisfying $(**)$ which is not left duo.

EXAMPLE. Let $\mathbb{H}$ be the Hamilton real quaternions. We will show that the polynomial ring $R = \mathbb{H}[x]$ is not left duo but SLB. Then Proposition 3.1. implies R satisfies $(**)$. Let L be any nonzero left ideal in R. Say, $s = \sum_{i=0}^{n} a_i x^i$ is a nonzero element in L. Consider $t = \left(\sum_{i=0}^{n} \overline{a}_i x^i\right)\left(\sum_{i=0}^{n} a_i x^i\right)$ where $\overline{a}_i$ is the conjugate $a_i \in \mathbb{H}$. Then t is a polynomial with coefficient in reals. So $Rt = tR$ is a two-sided ideal contained in L. Therefore R is a SLB. Next we consider the left ideal $R(x+i)$, $i^2 = -1$. With simple calculations, we can verify the left ideal $R(x+i)$ is not right ideal. Thus R is not left duo.

PROPOSITION 3.2. *If R satisfies $(**)$, then R is abelian.*

Proof. Let e be any idempotent in R. Then $Re \cap R(1-e) = 0$. So $(Re)(R(1-e)) = 0$ and $(R(1-e))Re = 0$. Thus $eR(1-e) = (1-e)Re = 0$, this shows $ex = xe$ for any x in R.

PROPOSITION 3.3. *Let R satisfy $(**)$. If R is a left nonsingular ring, then R is reduced.*

Proof. Suppose $a \in R$ and $a^2 = 0$. Then $Ra \subseteq \ell(a)$. By Zorn's lemma, there exists a complement left ideal K in R such that $Ra \oplus K$ is an essential left ideal in R. Since R satisfies $(**)$, $Ka \subseteq KRa = 0$. So $K \subseteq \ell(a)$, hence $Ra \oplus K \subseteq \ell(a)$. This implies $a \in Z_\ell(R) = 0$.

Using this result and a result in [7, Theorem 1], we can give a partial answer for Ming's question raised in [9, p. 157]. First we recall a result due to Ikeda and Nakayama [3] : R is a left p-injective ring iff every principal right ideal of R is a right annihilator.

PROPOSITION 3.4. *Let R satisfy $(**)$. Then the following are equivalent:*

(a) *R is a strongly regular ring.*
(b) *R is a semiprime ELT left p-injective ring.*

(c) *R is a left nonsingular, left p-injective ring.*

Proof. (a) $\Rightarrow$ (b) is well-known and (b) $\Rightarrow$ (c) is trivial. (c) $\Rightarrow$ (a): Proposition 3.3. and a result in [7, Theorem 1].

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