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RELATIONS BETWEEN THE ITÔ PROCESSES BASED ON THE WASSERSTEIN FUNCTION

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1. Introduction

Suppose that (S, d) is a Polish space and 0 is a fixed but arbitrarily chosen point in S. For each p with $1 \leq p < \infty$, define $\mathcal{M}_p = \mathcal{M}_p(S)$ to be the collection of all probability measures P on the Borel sets of S for which

$$\int_S d^p(X, 0) dP(X)$$

is finite.

Let P_1 and P_2 be members of $\mathcal{M}_p$ for $1 \leq p < \infty$ and μ‘ laws on $S \times S$ with marginals P_1 and P_2. The function defined by

$$W_p(P_1, P_2) = \left(inf_\mu \int d^p(X,Y) d\mu(X,Y) \right)^{\frac{1}{p}}$$

is called the L^p *Wasserstein metric* between P_1 and P_2 ([2, p.231]). A simple calculation shows that the Wasserstein functions W_p are metrics on the sets $\mathcal{M}_p$ for $1 \leq p < \infty$.

Let $S^+(\mathbf{R}^N)$ denote the space of nonnegative definite symmetric matrices. Given bounded measurable coefficient functions $a : [0, \infty) \to S^+(\mathbf{R}^N)$, $b : [0, \infty) \to \mathbf{R}^N$ and $\alpha : \mathbf{R}^N \to S^+(\mathbf{R}^N)$, $\beta : \mathbf{R}^N \to \mathbf{R}^N$, define the operator L by setting

$$L\big(a(x), b(x)\big) = \frac{1}{2} \sum_{i,j=1}^N a_{ij}(x) \partial x_i \partial x_j + \sum_{i=1}^N b_i(x) \partial x_i$$

$$L\big(\alpha(x), \beta(x)\big) = \frac{1}{2} \sum_{i,j=1}^N \alpha_{ij}(x) \partial x_i \partial x_j + \sum_{i=1}^N \beta_i(x) \partial x_i \,.$$

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Then we can define an operator $L\big(a(x,y), b(x,y)\big)$ on $\mathbf{R}^{2N}$ as coefficient functions

$$a(x,y) = \begin{pmatrix} a(x) & c(x,y) \\ c(x,y)^t & \alpha(y) \end{pmatrix}, \qquad b(x,y) = \begin{pmatrix} b(x) \\ \beta(y) \end{pmatrix}$$

where $c(x,y)$ is a real valued $N \times N$ matrix such that the matrix $a(x,y)$ is nonnegative definite. Throughout this note, the coefficients of all operators are assumed to be locally bounded and continuous. Moreover, we assume that the martingale problems for the diffusion processes are well-posed ([4, p.85]) and solutions for the operators $L\big(a(x), b(x)\big)$, $L\big(\alpha(x), \beta(x)\big)$ and $L\big(a(x,y), b(x,y)\big)$ are P_1, P_2 and μ, respectively.

In this note, we denote $y'(\cdot)$ to be the sample derivative of the y-process describing the position of the particles in both the strong (mean square) and pointwise senses and we say that it is the derived process. And we show that the derived process $y'(t)$ satisfies the stochastic differential equation and Itô process $x(t)$ ([1, p.166]) and $y(t)$ have same transition functions when time t runs infinite.

2. The Main Results

We begin with:

LEMMA 1. *Given y-process, there is a Brownian motion process $z(t)$ and derived process $y'(\cdot)$ such that*

$$dy'(T) = a^{\frac{1}{2}}\big(s+t, y(t), y'(t)\big)\,dz(t) + b\big(s+t, y(t), y'(t)\big)\,dt, \quad T \geq 0$$

where $a : [0,\infty) \times \mathbf{R}^N \to S^+(\mathbf{R}^N)$ and $b : [0,\infty) \times \mathbf{R}^N \to \mathbf{R}^N$.

Proof. Remembering, by Doob-Meyer's decomposition ([3, Theorem 6.12], [4, p.39]), there exists a unique natural integrable increasing process $\langle X\rangle(t)$ such that $X(t)^2 - \langle X\rangle(t)$ is a martingale. Let

$$Y(T) = \int_0^T a^{\frac{1}{2}}(s+t, y(t), y'(t))dy'(t), \quad Z(T) = \begin{pmatrix} y'(T) \\ Y(T) \end{pmatrix} \in \mathbf{R}^N \times \mathbf{R}^N$$

and

$$\begin{aligned} \langle \bar{Z}_i, \bar{Z}_j\rangle(dt) &= \langle\langle \bar{Z}, \bar{Z}\rangle\rangle(dt) \\ &= \begin{pmatrix} a(s+t, y(t), y'(t)) & 0 \\ 0 & I \end{pmatrix} dt \end{aligned}$$

where $\bar{Z}(T) = \begin{pmatrix} \bar{Y}(T) \\ Y(T) \end{pmatrix}$, $\bar{Y}(T) = y'(T) - \int_0^T b(s+t, y(t), y'(t))dt$ and I is an identity matrix. Define Π_1 and Π_2 to be the orthogonal projections of $\mathbf{R}^N$ onto range of a and $(a^{\frac{1}{2}})^t$, respectively. Set $\Pi_3 = I - \Pi_2$ and define by

$$z(T) = \int_0^T ((a^{\frac{1}{2}})^t \cdot \bar{a}, \Pi_3)(s+t, y(t), y'(t))d\bar{Z}(t)$$

where $a \cdot \bar{a} = \bar{a} \cdot a = \Pi_1$. Then

$$\langle\langle Z, Z\rangle\rangle(dt) = (\Pi_2 + \Pi_3)(s+t, y(t), y'(t))dt = Idt$$

and thus $z(t)$ is N-dimensional Brownian motion.
Therefore the result follows from the fact that

$$\bar{Y}(T) - y'(0) = \int_0^T d\bar{Y}(t) = \int_0^T a^{\frac{1}{2}}\left(s+t, y(t), y'(t)\right) dz(t)$$

with the choice of $z(t)$.

If $x(t)$ is a Itô process and $y(t)$ is the process given in the Lemma 1, let $P_1(t, x, \cdot)$ and $P_2(t, y, \cdot)$ be the transition functions of the marginal diffusion $x(t)$ and $y(t)$, respectively and we write

$$\rho(X, Y) = [\sum_{i=1}^{N}(X_i - Y_i)^p]^{\frac{1}{p}}.$$

We now meet:

THEOREM 2. *If there exist constants $R \geq 0, S > 0$ such that*

$$L\rho^p[x(t), y(t)] \leq R - S\rho^p[x(t), y(t)]\ ,$$

then

$$W_p[P_1(t, x, \cdot), P_2(t, y, \cdot)] \to 0 \quad as \ \ t \to \infty.$$

Proof. Write

$$S_N = inf\{t \geq 0 : |x(t) - y(t)| > N\} ,$$
$$T_R = inf\{t \geq 0 : |x(t)|^p + |y(t)|^p > R\}$$

and

$$s = S_N \wedge T_R .$$

Since the martingale problem is well-posed, we have

$$E\rho^p[x(t\wedge s), y(t\wedge s)] = \rho^p(x,y) + \int_0^t EL\rho^p[x(u\wedge s), y(u\wedge s)]\, du$$

and hence

$$\frac{d}{dt}E\rho^p[x(t\wedge s), y(t\wedge s)] = EL\rho^p[x(t\wedge s, y(t\wedge s)]$$
$$\leq R - SE\rho^p[x(t\wedge s), y(t\wedge s)] .$$

We therefore have

$$E\rho^p[x(t\wedge s), y(t\wedge s)] \leq E\rho^p\, [\, x(0 \wedge s),\, y(0 \wedge s)]e^{-St} + \int_0^t e^{-S(t-u)} R\, du :$$

writing $\rho\big(x(0), y(0)\big) = \rho(x,y)$, we thus have

$$E\rho^p[x(t\wedge s), y(t\wedge s)] \leq \frac{R}{S} + \rho^p(x,y)e^{-St} .$$

Letting $N \uparrow \infty$ and $R \uparrow \infty$, we obtain

$$E\rho^p[x(t), y(t)] \leq \frac{R}{S} + \rho^p(x,y)e^{-St} .$$

Since

$$\{E\rho^p[x(t), y(t)]\}^{\frac{1}{p}} = W_p[P_1(t,x,\cdot), P_2(t,y,\cdot)] ,$$

it follows that

$$W_p[P_1(t,x,\cdot), P_2(t,y,\cdot)] \leq \rho(x,y)e^{-\frac{St}{p}} .$$

Letting $t \to \infty$, the result follows.

Let $\mathcal{M}_\infty(S)$ be the set of all laws on S with bounded support and let P_1 and P_2 be members of $\mathcal{M}_\infty(S)$. The L^∞-distance is defined by

$$W_\infty(P_1, P_2) = inf_\mu \|d(X,Y)\|_\infty^{(\mu)} ,$$

where the subscription indicates that the usual L^∞-norm is taken with respect to μ and $d(X,Y)$ is in W_p-metric ([2, p.232]). Denote L^∞-metric by $l(X,Y)$.

We conclude with:

COROLLARY 3. *If P_1 and P_2 are finite, then*

$$W_\infty(P_1, P_2) \le l(x,y) \qquad \text{for all } t < \infty.$$

Proof. This at once follows from the Theorem 2.

REMARK. We note that if either P_1 or P_2 is infinite, then the Corollary 3 is false : for example, consider

$$d(x,y) = \begin{cases} 1 & \text{if } N \le x,y \le N + \frac{1}{N},\ N = 1,2,\cdots,\ x \ne y \\ 0 & \text{otherwise.} \end{cases}$$

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