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A DUAL LIMIT THEOREM IN A MEAN FIELD MODEL

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1. Introduction

Consider one particular problem of ferromagnetism in statistical mechanics. A ferromagnetic crystal can be considered to consist of n sites, where n is a large integer. The amount of magnetic spin will be denoted by $X_i^{(n)}, i = 1, 2, \ldots, n$. The magnetic spins at these n sites can be modeled by a triangular array of random variables $\{X_i^{(n)} : i = 1, 2, \ldots, n\} (n = 1, 2, \ldots)$. A standard model for the joint distribution of the spin random variables $(X_1^{(n)}, \ldots, X_n^{(n)})$ states that for any Borel set A in R^n

$$(1.1) \quad Pr[(X_1^{(n)}, \ldots, X_n^{(n)}) \in A] = d_n^{-1} \int_A \exp[-\beta \cdot H_n(x_1, \ldots, x_n)] \prod_{i=1}^{n} dP(x_i),$$

where d_n is a normalizing constant,P is a probability measure on R^1 and $\beta(> 0)$ is a constant which plays the role of inverse temperature. The function H_n is known as the Hamiltonian which represents the energy of the body at the configuration $(x_1, \ldots, x_n)$. The total magnetization present in the body is given by $S_n = \sum_{i=1}^n X_i^{(n)}$. When H_n takes the particular form $H_n(x_1, \ldots, x_n) = -(\sum x_i)^2/2n$, the model (1.1) is usually called the Curie-Weiss model in statistical mechanics literature.

In recent years, a number of results on the asymptotic distribution of $S_n = \sum_{i=1}^n X_i^{(n)}$ for this model have been established. Simon and Griffiths(1973) obtained the asymptotic distribution of S_n when P is the symmetric Bernoulli. Dunlop and Newman(1975) extended the result to

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the case where the spin random variables are vectors. In a two-paper series, Ellis and Newman(1978a, 1978b) extended the result of Simon and Griffiths to a larger class of probability distributions which contains the symmetric Bernoulli distribution. Jeon(1979) gave a simpler and statistically motivated proof of this result and Choi, Kim and Jeon(1989) obtained similar limit theorem for a wider class of Hamiltonians. The model considered by Choi, Kim and Jeon(1989) will be called the generalized Curie-Weiss model.

The purpose of this paper is to introduce the dual model of the generalized Curie-Weiss model and to establish a limit theorem for the dual model. The generalized Curie-Weiss model and its dual model are defined in Section 2 and the basic relationships between two models are given in Section 3. In Section 4, we state and prove our main result.

2. Generalized Curie-Weiss model and its dual model

In this section, we first define the generalized Curie-Weiss model and its dual model and then develop some notations and definitions necessary to describe our result. We also give some relationships between two models which play an important role in obtaining our main result.

For a given probability measure Q with $\phi_Q(t) = \int \exp(tx)dQ(x) < \infty$ for all real t, let L_Q be the class of probability measures P such that

$$\phi_P(t) = \int \exp(tx)\, dP(x) < \infty \qquad \text{for all } t \tag{2.1}$$

and

$$\int \phi_Q(x)\, dP(x) < \infty. \tag{2.2}$$

Let $\{X_i^{(n)} : i = 1, 2, \ldots, n\}(n = 1, 2, \ldots)$ be a triangular array with the joint distribution given by

$$d\mu_n(x_1, x_2, \ldots, x_n) = d_n^{-1} \cdot \exp[n\psi_Q(s_n/n)] \prod_{i=1}^{n} dP(x_i), \tag{2.3}$$

where $P \in L_Q$, $\psi_Q(t) = log\, \phi_Q(t)$ is the cumulant generating function of Q, d_n is a normalizing constant and $s_n = x_1 + \cdots + x_n$.

The model (2.3) defines the generalized Curie-Weiss model. See Choi, Kim and Jeon(1989) for more details on this model.

REMARK 2.1. By the convexity of ψ_Q, the probability distribution μ_n is well defined for each n. When Q is standard normal, the generalized Curie-Weiss model becomes the Curie-Weiss model in which case the condition $P \in L_Q$ is equivalent to that $\int \exp(x^2/2)dP(x) < \infty$. This is exactly the same as the condition considered by Ellis and Newman(1978b).

Let the distribution function F_Q of probability measure Q be such that

$$\begin{aligned} F_Q(x) &= 0, & x &< a, \\ 0 < F_Q(x) &< 1, & a &< x < b, \\ F_Q(x) &= 1, & b &< x, \end{aligned}$$

and let $D_Q = (a, b)$, where if desired $a = -\infty$ or $b = \infty$, or both.

For given probability measures Q and $P(\in L_Q)$, define

$$G_{QP}(u) = \gamma_Q(u) - \psi_P(u) \qquad \text{for all } u, \tag{2.4}$$

where $\gamma_Q(u) = \sup_s[us - \psi_Q(s)]$ is the large deviation rate of Q and $\psi_P(u) = \log \phi_P(u)$ is the cumulant generating function of P.

REMARK 2.2. If Q is the standard normal, $G_{QP}(u) = u^2/2 - \psi_P(u)$, $u \in R$. This is exactly the same as the function studied by Ellis and Newman (1987b) in the Curie-Weiss model. The function G_{QP} plays an important role in determining the asymptotic behavior of the total magnetization S_n for the generalized Curie-Weiss model.

DEFINITION 2.1. A real number $m(\in D_Q)$ is said to be a global minimum for G_{QP} if

$$G_{QP}(u) \geq G_{QP}(m) \qquad \text{for} \quad \text{all} \quad u \in D_Q \, .$$

DEFINITION 2.2. A global minimum m for G_{QP} is said to be of type k if

$$G_{QP}(m+u) - G_{QP}(m) = \frac{c_{2k}u^{2k}}{(2k)!} + o(u^{2k}) \qquad \text{as } u \to 0 \, , \tag{2.5}$$

where $c_{2k} = G_{QP}^{(2k)}(m)$ is strictly positive.

When $D_Q = (-\infty, \infty)$, We have the following lemma due to Choi, Kim and Jeon(1989).

LEMMA 2.1. *For a given probability measure Q, let $P \in L_Q$. Let $D_Q = (-\infty, \infty)$. Then $G_{QP}(t) \to \infty$ as $|t| \to \infty$. Thus, G_{QP} has only a finite number of global minima.*

In the generalized Curie-Weiss model, we first fix a probability measure Q. Then the model is defined for $P \in L_Q$. Interchanging the roles of P and Q in the model (2.3), we define the dual of the generalized Curie-Weiss model as

$$d\mu_n^*(x_1, x_2, \ldots, x_n) = D_n^{-1} \cdot \exp[n\psi_P(s_n/n)] \prod_{i=1}^{n} dQ(x_i), \tag{2.6}$$

where D_n is a normalizing constant and ψ_P is the cumulant generating function of probability measure P.

REMARK 2.3. The dual model is also well defined by the similar reason stated in Remark 2.1 and the following lemma.

LEMMA 2.2. *For a given probability measure Q, let $P \in L_Q$. Then $Q \in L_P$.*

Proof. By the Fubini's theorem,

$$\begin{aligned} \int \phi_P(y)\, dQ(y) &= \int \int \exp(y \cdot x)\, dP(x)\, dQ(y) \\ &= \int \phi_Q(x)\, dP(x) < \infty. \end{aligned}$$

For the dual model, let $F_P(x)$, the distribution function of P, be such that

$$\begin{aligned} &F_P(x) = 0, && x < c, \\ &0 < F_P(x) < 1, && c < x < d, \\ &F_P(x) = 1, && d < x, \end{aligned}$$

and let $D_P = (c, d)$, where $-\infty \le c < d \le \infty$.
The function corresponding to G_{QP} of the original model is defined by

$$G_{PQ}(u) = \gamma_P(u) - \psi_Q(u) \quad \text{for all} \quad u, \tag{2.7}$$

where $\gamma_P(u) = \sup_s [us - \psi_P(s)]$ is the large deviation rate of P.

3. Some relationships between generalized Curie-Weiss model and its dual.

In this section, we study some properties of γ_Q, the large deviation rate of probability measure Q, and give the relationship between the functions G_{QP} and G_{PQ} defined in (2.4) and (2.7) respectively. We first state the lemma due to Daniels(1954).

LEMMA 3.1. *Let the distribution function $F(x)$ be such that $F(x) = 0$ for $x < \alpha$, $0 < F(x) < 1$ for $\alpha < x < \beta$, $F(x) = 1$ for $\beta < x$, where if desired $\alpha = -\infty$ or $\beta = \infty$, or both. Suppose that the moment generating function of F converges for all t, i.e.,*

$$\phi(t) = \exp\{\psi(t)\} = \int_{-\infty}^{\infty} e^{tx}\, dF(x) < \infty \qquad \text{for all } t.$$

Then, for every ξ in $\alpha < \xi < \beta$, there exists a unique solution t_0 of $\psi'(t) = \xi$. And as t increases from $-\infty$ to ∞, $\psi'(t)$ increases continuously from $\xi = \alpha$ to $\xi = \beta$.

Let $P \in L_Q$. Then $Q \in L_P$ by Lemma 2.2. Thus, by Lemma 3.1, for each $u \in D_Q$, $\psi'_Q(t) = u$ has a unique solution and, for each $u \in D_P$, $\psi'_P(t) = u$ also has a unique solution. Furthermore we have the following useful equations:
for any $u \in D_Q$,

$$\gamma_Q(u) = u \cdot \psi_Q'^{-1}(u) - \psi_Q(\psi'^{-1}_Q(u)) \tag{3.1}$$

and

$$\gamma'_Q(u) = \psi'^{-1}_Q(u). \tag{3.2}$$

Note that (3.1) and (3.2) still hold for $\gamma_P(u)$ and $\gamma'_P(u)$ if we replace the subscript Q by P.

LEMMA 3.2. *Let $G_{QP}(u)$ and $G_{PQ}(u)$ be defined as in (2.4) and (2.7). Then we have*

$$\begin{aligned} G_{QP}(u) = &(G_{PQ} \circ \psi'_P)(u) + (\psi_Q \circ \psi'_P)(u) \\ &- (\psi_Q \circ \psi'^{-1}_Q)(u) + u \cdot G'_{QP}(u), \quad u \in D_Q, \end{aligned} \tag{3.3}$$

and

$$(3.4)\qquad \begin{aligned} G_{PQ}(u) =&(G_{QP}\circ\psi'_Q)(u)+(\psi_P\circ\psi'_Q)(u)\\ &-(\psi_P\circ\psi'^{-1}_P)(u)+u\cdot G'_{PQ}(u),\quad u\in D_P. \end{aligned}$$

Proof. We have only to prove (3.4) by symmetry. For any $u\in D_P$,

$$\begin{aligned} G_{PQ}(u) =&\gamma_P(u)-\psi_Q(u)\\ =&u\cdot\psi'^{-1}_P(u)-(\psi_P\circ\psi'^{-1}_P)(u)-\psi_Q(u) \qquad \text{by (3.1)}\\ =&u\cdot\psi'_Q(u)-\psi_Q(u)-(\psi_P\circ\psi'_Q)(u)+(\psi_P\circ\psi'_Q)(u)\\ &-(\psi_P\circ\psi'^{-1}_P)(u)+u\cdot\psi'^{-1}_P(u)-u\cdot\psi'_Q(u). \end{aligned}$$

Since $[(I\times\psi'^{-1}_Q)\circ\psi'_Q](u)=u\cdot\psi'_Q(u)$, where $I(x)=x$, we can write, by (3.1),

$$\begin{aligned} u\cdot&\psi'_Q(u)-\psi_Q(u)-(\psi_P\circ\psi'_Q)(u)\\ &=[(I\times\psi'^{-1}_Q-\psi_Q\circ\psi'^{-1}_Q-\psi_P)\circ\psi'_Q](u)\\ &=(G_{QP}\circ\psi'_Q)(u) \end{aligned}$$

and, by (3.2), $G'_{PQ}(u)=\psi'^{-1}_P(u)-\psi'_Q(u)$ for each $u\in D_P$.

THEOREM 3.1. *Let m be a global minimum for G_{QP}. Then $\psi'^{-1}_Q(m) = \psi'_P(m)$ ($=m^D$, say) and m^D is a global minimum for G_{PQ}. Conversely if m^D is a global minimum for G_{PQ}, then $\psi'^{-1}_P(m^D)=\psi'_Q(m^D)$ (=m, say) and m is a global minimum for G_{QP}. Furthermore $G_{QP}(m) = G_{PQ}(m^D)$.*

Proof. By symmetry we prove only the first case. Assume that m is a global minimum for G_{QP}. Then, since $G'_{QP}(u)=\psi'^{-1}_Q(u)-\psi'_P(u)$ by (3.2), $G'_{QP}(m)=\psi'^{-1}_Q(m)-\psi'_P(m)=0$. Thus $\psi'^{-1}_Q(m)=\psi'_P(m)=m^D$, say. Since $\psi'_Q(m^D)=m$, $G_{QP}(m)=G_{PQ}(m^D)$ is obtained immediately by Lemma 3.2.

Now, we will prove that m^D is a global minimum for G_{PQ}. By Lemma 3.2 and Taylor's expansion, for any $u\in D_P$, we have

(3.5)

$$\begin{aligned} G_{PQ}(u) =&(G_{QP}\circ\psi'_Q)(u)+(\psi_P\circ\psi'_Q)(u)-(\psi_P\circ\psi'^{-1}_P)(u)+u\cdot G'_{PQ}(u)\\ =&(G_{QP}\circ\psi'_Q)(u)+\psi''_P[\psi'^{-1}_P(u)-\theta\cdot G'_{PQ}(u)]\cdot(G'_{PQ}(u))^2/2, \end{aligned}$$

where $0 < \theta < 1$.

By the assumption that m is a global minimum and by (3.5), we have, for any $u \in D_P$,

$$(3.6) \qquad G_{PQ}(m^D) = G_{QP}(m) \leq (G_{QP} \circ \psi'_Q)(u) \leq G_{PQ}(u).$$

This completes the proof of Theorem 3.1.

THEOREM 3.2. *Assume that G_{QP} has the unique global minimum of type k at m and*

$$(3.7) \qquad \inf_{u \in D_Q} G_{QP}(u) < \min\left\{\liminf_{u \to a} G_{QP}(u), \liminf_{u \to b} G_{QP}(u)\right\}.$$

Then G_{PQ} has the unique global minimum of type k at $m^D = \psi'_P(m)$ and in this case $G_{PQ}^{(2k)}(m^D) = c_{2k}[\psi''_P(m)]^{-2k}$, where $c_{2k} = G_{QP}^{(2k)}(m)$. Furthermore

$$(3.8) \qquad \inf_{u \in D_P} G_{PQ}(u) < \min\left\{\liminf_{u \to c} G_{PQ}(u), \liminf_{u \to d} G_{PQ}(u)\right\}.$$

Proof. Since m is the unique global minimum, the inequalities in (3.6) hold strictly for $u \neq m^D$ and $u \in D_P$. Hence $m^D = \psi'_P(m)$ is the unique global minimum for G_{PQ}. Since G_{QP} has the unique global minimum of type k at m,

$$(3.9) \qquad G_{QP}(m+u) - G_{QP}(m) = \frac{c_{2k} \cdot u^{2k}}{(2k)!} + o(u^{2k}) \quad \text{as } u \to 0$$

and

$$(3.10) \qquad G'_{QP}(m+u) = \frac{c_{2k} \cdot u^{2k-1}}{(2k-1)!} + o(u^{2k-1}) \quad \text{as } u \to 0,$$

where $c_{2k} = G_{QP}^{(2k)}(m) > 0$.
And by the duality of G_{QP} and G_{PQ} and by (3.6), for any $u \in D_Q$,

$$(3.11) \quad (G_{PQ} \circ \psi'_P)(u) \\ = G_{QP}(u) - \psi''_Q[\psi'^{-1}_Q(u) - \eta \cdot G'_{QP}(u)](G'_{QP}(u)/2)^2,$$

where $0 < \eta < 1$.
From (3.9), (3.10) and (3.11), we have

(3.12)

$$\begin{aligned}
&(G_{PQ} \circ \psi'_P)(m+u) - G_{PQ}(m^D) \\
&= G_{QP}(m+u) - G_{QP}(m) - \psi''_Q[\psi'^{-1}_Q(m+u) - \eta\{\psi'^{-1}_Q(m+u) \\
&\quad - \psi'_P(m+u)\}] \times (G'_{QP}(m+u))^2/2 \\
&= \frac{c_{2k} \cdot u^{2k}}{(2k)!} - \psi''_Q(m^D) \cdot \frac{(c_{2k})^2 \cdot u^{2(2k-1)}}{\{(2k-1)!\}^2 \cdot 2} + o(u^{2k}) \qquad \text{as } u \to 0 \\
&= \frac{c_{2k} \cdot u^{2k}}{(2k)!} + o(u^{2k}) \qquad \text{as } u \to 0.
\end{aligned}$$

Since $\psi'^{-1}_P(m^D + u) - \psi'^{-1}_P(m^D) = (\psi'^{-1}_P)'(m^D) \cdot u + o(u)$ as $u \to 0$, we have

$$\begin{aligned}
&G_{PQ}(m^D + u) - G_{PQ}(m^D) \\
&= (G_{PQ} \circ \psi'_P)(\psi'^{-1}_P(m^D + u)) - G_{PQ}(m^D) \\
&= \frac{c_{2k}[(\psi'^{-1}_P)'(m^D)u + o(u)]^{2k}}{(2k)!} + o(u^{2k}) \quad \text{as} \quad u \to 0 \qquad \text{by (3.12)} \\
&= \frac{c_{2k}[\psi''_P(m)]^{-2k}u^{2k}}{(2k)!} + o(u^{2k}) \quad \text{as} \quad u \to 0 \\
&= \frac{c'_{2k}u^{2k}}{(2k)!} + o(u^{2k}) \quad \text{as} \quad u \to 0,
\end{aligned}$$

where $c'_{2k} = G^{(2k)}_{PQ}(m^D) = c_{2k}[\psi''_P(m)]^{-2k}$.

Finally, we prove the inequality (3.8). By Lemma 3.1 and Theorem 3.1, there exist ξ_1 and $\xi_2 (a \le \xi_1 < m < \xi_2 \le b)$ such that $\lim_{u \downarrow c} \psi'_Q(u) = \xi_1$ and $\lim_{u \uparrow d} \psi'_Q(u) = \xi_2$.
By assumption (3.7) and by the fact that $G_{QP}(m) = \inf_{u \in D_Q} G_{QP}(u)$ is

the unique global minimum, we have

$$\begin{aligned} G_{PQ}(m^D) = G_{QP}(m) &< \min\{\liminf_{t\to\xi_1} G_{QP}(t),\ \liminf_{t\to\xi_2} G_{QP}(t)\} \\ &= \min\{\liminf_{u\to c} G_{QP}[\psi'_Q(u)], \liminf_{u\to d} G_{QP}[\psi'_Q(u)]\} \\ &< \min\{\liminf_{u\to c} G_{PQ}(u), \liminf_{u\to d} G_{PQ}(u)\} \quad \text{by}(3.5). \end{aligned}$$

4. Main result

Let the distribution function F and the moment generating function $\phi(t)$ of F be such as in Lemma 3.1. Further let $\psi(t)$ and $\gamma(t)$ be the cumulant generating function and the large deviation rate of F respectively.

Let $\{Y_n\}$ be a sequence of independent and identically distributed random variables with common distribution function F and f_n be the probability density function of $\sum_{i=1}^n Y_i/n$.

For certain classes of densities, Daniels(1954) proved the uniform local limit theorem for $\sum_{i=1}^n Y_i/n$ as follows ;

$$f_n(x) = \sqrt{\frac{n}{2\pi}}\sigma^{-1}(t) \cdot \exp[-n\gamma(x)][1+o(1)] \quad \text{as} \quad n\to\infty \tag{4.1}$$

holds uniformly in $x \in D = (\alpha,\beta) = \{\psi'(t)|t\in R\}$, where $\psi'(t) = x$ and $\sigma^2(t) = \psi''(t)$.

Now we will state our main result.

THEOREM 4.1. *Let P and Q be probability measures which satisfy the condition (4.1). Let $\{X_1^{(n)}, X_2^{(n)},\ldots,X_n^{(n)}\}$ and $\{X_1^{D(n)}, X_2^{D(n)},\ldots, X_n^{D(n)}\}$ be triangular arrays of dependent and identically distributed random variables with joint distributions $d\mu_n$ and $d\mu_n^*$ given by (2.3) and (2.6) respectively. Assume that G_{QP} has the unique global minimum of type k at m and also assume*

$$\inf_{u\in D_Q} G_{QP}(u) < \min\{\liminf_{u\to a} G_{QP}(u), \liminf_{u\to b} G_{QP}(u)\}. \tag{4.2}$$

Then

$$(4.3)\qquad \frac{S_n - nm^D}{m_1^D n^{1-\frac{1}{2k}}} \xrightarrow{d} \begin{cases} N(0, \frac{1}{m_1^D} + \frac{1}{c_2}), & \text{if } k = 1 \\ \exp\{-c_{2k}z^{2k}/(2k)!\}, & \text{if } k \geq 2, \end{cases}$$

and

$$(4.4)\qquad \frac{S_n^D - nm}{m_1 n^{1-\frac{1}{2k}}} \xrightarrow{d} \begin{cases} N(0, \frac{1}{m_1} + \frac{1}{c_2'}), & \text{if } k = 1 \\ \exp\{-c_{2k}' z^{2k}/(2k)!\}, & \text{if } k \geq 2, \end{cases}$$

where $m^D = \psi_P'(m)$, $m = \psi_Q'(m^D)$, $m_1^D = \psi_P''(m)$, $m_1 = \psi_Q''(m^D)$, $c_{2k} = G_{QP}^{(2k)}(m)$, $c_{2k}' = G_{PQ}^{(2k)}(m^D)$, $S_n = \sum_{i=1}^n X_i^{(n)}$ *and* $S_n^D = \sum_{i=1}^n X_i^{D(n)}$.

Proof. Since Q satisfies the condition (4.1), we can express the joint distribution μ_n as follows ;

$$\begin{aligned}
& d\mu_n(x_1, x_2, \ldots, x_n) \\
&= d_n^{-1} \exp\left\{n\psi_Q\left(\sum x_i/n\right)\right\} \prod_{i=1}^n dP(x_i) \\
&= d_n^{-1} \left[\int \exp\left\{\left(\sum x_i\right)u\right\} f_n(u)du\right] \prod_{i=1}^n dP(x_i) \\
&= d_n^{-1} n^{-\frac{1}{2k}} \int \left[\prod_{i=1}^n \exp\{x_i(m + zn^{-\frac{1}{2k}})\}dP(x_i)\right] f_n(m + zn^{-\frac{1}{2k}})dz \\
&= d_n^{-1}(2\pi)^{-\frac{1}{2}} n^{\frac{k-1}{2k}} \\
&\qquad \times \int \left[\prod_{i=1}^n \exp\{x_i(m + zn^{-\frac{1}{2k}}) - \psi_P(m + zn^{-\frac{1}{2k}})\}dP(x_i)\right] \\
&\qquad \times \exp\left[-nG_{QP}(m + zn^{-\frac{1}{2k}})\right] \sigma^{-1}(t_n)[1 + o(1)]dz,
\end{aligned}$$

where $\psi_Q'(t_n) = m + zn^{-\frac{1}{2k}}$ and $\sigma^2(t_n) = \psi_Q''(t_n)$.
Thus we have

$$(4.5)\qquad d\mu_n(x_1, x_2, \ldots, x_n) = K_n^{-1} \int \prod_{i=1}^n dM_{n,z}(x_i) \cdot h_n(z)dz,$$

where

$$dM_{n,z}(x) = \exp\left[x(m+zn^{-\frac{1}{2k}}) - \psi_P(m+zn^{-\frac{1}{2k}})\right]dP(x),$$

$$h_n(z) = \exp\left[-n\{G_{PQ}(m+zn^{-\frac{1}{2k}}) - G_{PQ}(m)\}\right]\sigma^{-1}(t_n)[1+o(1)],$$

and $$K_n = d_n(2\pi)^{\frac{1}{2}}n^{-\frac{k-1}{2k}}\exp\{nG_{QP}(m)\}.$$

Since $\int_{R^n} d\mu_n(x_1, x_2, \ldots, x_n) = 1$ and $\int dM_{n,z}(x_i) = 1$ for each i, we have $K_n = \int h_n(z)dz$. Thus $h_n^*(z) = K_n^{-1}h_n(z)$ is a probability density function for each n.

Now, as $n \to \infty$

$$\begin{aligned}
&\log E_{M_{n,z}}\left[\exp\{tn^{-(1-\frac{1}{2k})}S_n\}\right] \\
&= n\left\{\psi_P(tn^{-(1-\frac{1}{2k})} + m + zn^{-\frac{1}{2k}}) - \psi_P(m+zn^{-\frac{1}{2k}})\right\} \\
&= n\left\{\psi_P'(m+zn^{-\frac{1}{2k}})tn^{-(1-\frac{1}{2k})} + \frac{1}{2}\psi_P''(m+zn^{-\frac{1}{2k}})t^2n^{-(2-\frac{1}{k})} + o(n^{-1})\right\} \\
&= n^{\frac{1}{2k}}t\psi_P'(m) + \psi_P''(m)tz + \frac{1}{2}n^{-(1-\frac{1}{k})}t^2\psi_P''(m) + o(1)
\end{aligned}$$

This shows that, under $M_{n,z}$,

$$\frac{S_n - nm^D}{m_1^D n^{1-\frac{1}{2k}}} \xrightarrow{d} \begin{cases} N(z, \frac{1}{m_1^D}), & \text{if} \quad k=1 \\ \delta(s-z), & \text{if} \quad k \geq 2 \end{cases} \tag{4.6}$$

where $m^D = \psi_P'(m), m_1^D = \psi_P''(m)$ and $\delta(x - x_0)$ is a point mass at x_0.

The representation (4.5) of $d\mu_n$ shows that we can introduce a random variable V_n with the probability density function $h_n^*(z)$ such that , given $V_n = z$, the $X_i^{(n)'}s$ are independent and identically distributed random variables with the distribution $dM_{n,z}(x)$. It is now easy to obtain the limiting distribution of $(S_n - nm^D)/m_1^D n^{1-\frac{1}{2k}}$, given $V_n = z$, by showing that the moment generating function converges to a moment generating function as described above.

By Lemma 3.3 and Lemma 3.4 of Choi, Kim and Jeon(1989), we have, as $n \to \infty$,

$$(4.7) \qquad h_n^*(z) \longrightarrow h^*(z) \quad \text{for each } z$$

where $h^*(z) = \exp\{-c_{2k}z^{2k}/(2k)!\}/\int \exp\{-c_{2k}z^{2k}/(2k)!\}dz$.

By applying Theorem 2.1 of Sethuraman(1961), (4.6) and (4.7), the proof of (4.3) is completed.

Since P also satisfies the condition (4.1), we also obtain (4.4) by Theorem 3.2 and the proof is completed.

REMARK 4.1. In the special case $D_Q = (-\infty, \infty)$, we can dispense with the assumption (4.2). In this case the assumption (4.2) is automatically satisfied by Lemma 2.1.

5. Example

Let Q be the standard normal distribution and P be the triangular distribution on (-2b, 2b) with $b = \sqrt{3/2}$. Then the models (2.3) and (2.6) become, respectively,

$$(5.1) \qquad d\mu_n(x_1, x_2, \ldots, x_n) = d_n^{-1} \exp\{s_n^2/2n\} \prod_{i=1}^{n} dP(x_i)$$

and

$$(5.2) \qquad d\mu_n^*(x_1, x_2, \ldots, x_n) = D_n^{-1} \left[\frac{n \sinh(bs_n/n)}{bs_n}\right]^{2n} \prod_{i=1}^{n} dQ(x_i).$$

Clearly $P \in L_Q$ and the uniformity condition (4.1) for Q holds. Since $D_Q = (-\infty, \infty)$, the condition (4.2) is satisfied [see Remark 4.1]. The uniformity condition (4.1) for P was also shown to be satisfied by Daniels(1954) [see Example 7.4 there].

Since $\gamma_Q(u) = u^2/2$ and $\psi_P(u) = 2log[sinh(bu)/bu]^2, G_{QP}$ is even function and $G_{QP}(0) = 0$. And since $G'_{QP}(u) = u - 2[\ coth(bu) \cdot b - 1/u\ ] > 0$, for all $u > 0$, $m = 0$ is the unique global minimum for G_{QP}. Furthermore it can be shown by simple calculation that

$$G'_{QP}(0) = \psi'_P(0) = 0,$$

$$\begin{aligned} G''_{QP}(0) &= 1-\psi''_P(0) = 0, \\ G'''_{QP}(0) &= -\psi'''_P(0) = 0, \\ G^{(4)}_{QP}(0) &= -\{\phi^{(4)}_P(0)-3\} = 3/5 > 0. \end{aligned}$$

Thus G_{QP} has the unique global minimum $m=0$ of type 2. By Theorem 3.2, G_{PQ} also has the unique global minimum of type 2 at $m^D = \psi'_P(0) = 0$ and $c'_4 = G^{(4)}_{PQ}(0) = G^{(4)}_{QP}(0)[\psi''_P(0)]^{-4} = 3/5$. Since $m^D_1 = \psi''_P(0) = 1$ and $m_1 = \psi''_Q(0) = 1$, we have , by Theorem 4.1,

$$S_n/n^{\frac{3}{4}} \xrightarrow{d} \exp\{-z^4/40\} \quad \text{and} \quad S^D_n/n^{\frac{3}{4}} \xrightarrow{d} \exp\{-z^4/40\}.$$

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