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THE INVARIANCE PRINCIPLE FOR TWO-PARAMETER ASSOCIATED PROCESSES

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1. Introduction and notations

Let $\{X_{(j_1,j_2)} : j_1, j_2 \in N\}$ be a two parameter array of random variables on some probability space $(\Omega, \mathcal{F}, P)$ with $EX_{(j_1,j_2)} = 0$, $EX^2_{(j_1,j_2)} < \infty$. For $n \in N$, put

$$S_{(n,n)} = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} X_{(j_1,j_2)}, \quad \sigma^2_{(n,n)} = ES^2_{(n,n)}.$$

$\{X_{(j_1,j_2)}\}$ is said to satisfy the central limit theorem if $\sigma^{-1}_{(n,n)} S_{(n,n)}$ is asymptotically normally distributed as $n \to \infty$. Define

$$W_n(t_1,t_2) = \sigma^{-1}_{(n,n)} S_{([nt_1],[nt_2])}, t_1, t_2 \in [0,1].$$

Then W_n is a measurable map $(\Omega, \mathcal{F})$ into $(\mathcal{D}_2, \mathcal{B}(\mathcal{D}_2))$, where $\mathcal{D}_2$ is the space of all functions on $[0,1]^2$ which have left hand limits and are continuous from the right and $\mathcal{B}(\mathcal{D}_2)$ is the Borel σ-field induced by the Skorohod topology.

If W_n converges weakly to the two parameter Wiener process W on $\mathcal{D}_2$, we write $W_n \to {}_n W$ and say that $\{W_{(j_1,j_2)} : j_1, j_2 \in Z^2\}$ fulfills the invariance principle.

A finite collection $\{X_1, \dots, X_m\}$ of random variables is associated if for any two coordinatewise nondecreasing functions f_1, f_2 on R^m such that $\tilde{f}_i = f_i(X_1, \dots, X_m)$ has finite variance for $i = 1, 2$, there holds $\mathrm{Cov}(\tilde{f}_1, \tilde{f}_2) \geq 0$. An infinite collection is associated if every finite subcollection is associated ([1],[2]). Newman (1980, 1984) and Newman and

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Wright (1981, 1982) have concerned with limit theorems for a sequence of associated random variables. In nonstationary case, Cox and Grimmett (1984) proved the central limit theorem for associated random variables by certain conditions on the moments of random variables, and Birkel (1988) studied the invariance principle for one parameter associated processes.

Our result is an extension of Birkel's invariance principle (1988) to the two parameter case and an extension of Newman and Wright's invariance principle (1982) to the nonstationary case.

In section 2, we obtain two lemmas which may be regarded as extensions of Lemmas 1 and 2 of Birkel (1988) to two parameter case. These lemmas can be extended to a general d-dimension ($d > 2$). In Section 3, we obtain a theorem similar to Theorem 15.5 of Billingsley (1968) for two parameter associated processes and prove a general invariance principle for two parameter associated process which requires neither stationarity nor the finiteness of $u(r)$($u(r)$ is the covariance coefficient which will be specified later). In place of that we assume a condition on the covariance, namely

$$\sigma^{-2}_{(n,n)}\text{Cov}(S_{(nk_1,nk_2)}, S_{(nl_1,nl_2)}) \to k_1k_2, \text{ for } k_i \leq l_i \in N,\ i = 1,2.$$

Our results also seem to be applicable to the associated random measures, specifically to Poisson cluster random measure (cf. [4], [8]).

2. Preliminaries

Cox and Grimmett (1984) obtained the following central limit theorem using the coefficient

$$u(r) = \sup_{\underline{k}\geq 1} \sum_{\underline{j}:\|\underline{j}-\underline{k}\|\geq r} \text{Cov}(X_{\underline{j}}, X_{\underline{k}}), \quad r \in N \cup \{0\},$$

where $\|\underline{k} - \underline{j}\| = \max(|k_1 - j_1|, |k_2 - j_2|)$, $\underline{k} = (k_1, k_2)$, $\underline{j} = (j_1, j_2)$.

THEOREM 2.1. (Cox and Grimmett (1984)) *Let $\{X_{(j_1,j_2)} : j_1, j_2 \in N\}$ be a two-parameter array of associated random variables with*

$EX_{(j_1,j_2)} = 0$, $EX^2_{(j_1,j_2)} < \infty$. *Assume*

$$u(r) \to 0, \quad u(0) < \infty, \tag{2.1}$$

$$\inf_{j_1,j_2 \in N} \mathrm{Var}\, X_{(j_1,j_2)} > 0, \tag{2.2}$$

$$\sup_{j_1,j_2 \in N} E|X_{(j_1,j_2)}|^3 < \infty. \tag{2.3}$$

Then $\sigma^{-1}_{(n,n)} S_{(n,n)}$ *is asymptotically normally distributed.*

The following lemmas are regarded as extensions of Lemmas 1 and 2 of Birkel [3] to the two parameter case.

LEMMA 2.2. *Let* $\{X_{(j_1,j_2)} : j_1, j_2 \in N\}$ *be a two-parameter array of associated random variables with* $EX_{(j_1,j_2)} = 0$, $EX^2_{(j_1,j_2)} < \infty$. *Then the following conditions are equivalent:*

(i) $\sigma^{-2}_{(n,n)} \sigma^2_{(nk_1,nk_2)} \to k_1 k_2$ *for* $k_i \in N$, $i = 1, 2$,

(ii) $\sigma^{-2}_{(n,n)} \sigma^2_{([nt_1],[nt_2])} \to t_1 t_2$ *for* $t_1, t_2 > 0$.

Proof. This can be accomplished by the technique similar to Lemma 1 of Birkel [3].

LEMMA 2.3. *Let* $\{X_{\underline{j}} : \underline{j} \geq \underline{1}\}$ *be a 2-parameter array of associated random variables with* $EX_{\underline{j}} = 0$, $EX^2_{\underline{j}} < \infty$. *Assume that* (i) *of Lemma 2.2 is fulfilled. Then the following conditions are equivalent:*

(i) $\sigma^{-2}_{(n,n)} E\{(S_{n\underline{j}} - S_{n\underline{i}})(S_{n\underline{l}} - S_{n\underline{k}})\} \to 0$ *for* $\underline{0} \leq \underline{i} \leq \underline{j} \leq \underline{k} \leq \underline{l} \in Z^2$,

(ii) $\sigma^{-2}_{(n,n)} E\{S_{n\underline{k}} S_{n\underline{l}}\} \to |\underline{k}|$ *for* $\underline{0} \leq \underline{k} \leq \underline{l} \in Z^2$,

(iii) $\sigma^{-2}_{(n,n)} E\{S_{n\underline{l}} S_{[n\underline{t}]}\} \to t_1 t_2$ *for* $t_i \in [0, 1]$,

(iv) $\sigma^{-2}_{(n,n)} E\{(S_{[n\underline{t}]} - S_{[n,\underline{s}]})(S_{[n\underline{v}]} - S_{[n\underline{u}]})\} \to 0$ *for* $\underline{0} \leq \underline{s} \leq \underline{t} \leq \underline{u} \leq \underline{v}$,

(v) $\sigma^{-2}_{(n,n)} E\{(S_{[n\underline{t}]} - S_{[n\underline{s}]})(S_{[n\underline{v}]} - S_{[n\underline{u}]})\} \to 0$ *for* $\underline{0} \leq \underline{s} \leq \underline{t} \leq \underline{u} \leq \underline{v} \leq \underline{1}$.

Proof. Using Lemma 2.2 in this paper, the proof is obtained by the similar arguments to those of Lemma 2 of Birkel [3].

REMARK. Note that Lemmas 2.2 and 2.3 can be extended to the d-parameter case ($d > 2$) without any more assumptions.

3. Main results

THEOREM 3.1. *Let $\{X_{(j_1,j_2)} : j_1, j_2 \in N\}$ be a two-parameter array of associated random variables with $EX_{(j_1,j_2)} = 0$, $EX^2_{(j_1,j_2)} < \infty$. Assume*

$$(3.1) \quad \sigma^{-2}_{(n,n)} E(S_{(nk_1,nk_2)} S_{(nl_1,nl_2)}) \to k_1 k_2, \ \text{for } k_i \le l_i \in N, \ i = 1,2,$$

$$(3.2) \quad \{\sigma^{-2}_{(n,n)} (S_{(n+m_1,n+m_2)} - S_{(m_1,m_2)})^2 : m_1, m_2 \in N \cup \{0,\}, \ n \in N\}$$

is uniformly integrable. Then for every $\epsilon > 0$, we have

$$(3.3) \qquad \lim_{\delta\to 0} \lim_{n\to\infty} \sup P\{w(W_n, \delta) \ge \epsilon\} = 0,$$

where $w(W_n,\delta) = \sup_{\|\underline{s}-\underline{t}\|\le\delta} |W_n(\underline{s}) - W_n(\underline{t})|$, $\underline{s} = (s_1, s_2)$, $\underline{t} = (t_1, t_2)$, and $\|\underline{s} - \underline{t}\| = \max(|s_1 - t_1|, |s_2 - t_2|)$.

Proof. Condition (3.1) implies (i) of Lemma 2.2 and hence, by Lemma 2.2,

$$(3.4) \qquad \sigma^{-2}_{(n,n)} \sigma^2_{([nt_1],[nt_2])} \to t_1 t_2 \quad \text{for} \quad t_1, t_2 > 0.$$

The relation (40) in Theorem 10 of Newman and Wright (1982) yields

(3.5)

$$\begin{aligned} &P\{w(W_n,\delta) > \epsilon\} \\ &\le P\{\max_{\substack{[nt_1]<r_1\le[n(t_1+\delta)] \\ [nt_2]<r_2\le[n(t_2+\delta)]}} |S_{(r_1,r_2)} - S_{([nt_1],[nt_2])}| \ge \sigma_{(n,n)}\frac{\epsilon}{9}\} \\ &\le 3^{\frac{3}{2}} 2^{-\frac{1}{4}} (E(S_{([n(t_1+\delta)],[n(t_2+\delta)])} - S_{([nt_1],[nt_2])})^2 (18)^2 (\sigma_{(n,n)}\epsilon)^{-2})^{\frac{3}{4}} \cdot \\ &\quad (P\{|S_{([n(t_1+\delta)],[n(t_2+\delta)])} - S_{([nt_1],[nt_2])}| \ge \sigma_{(n,n)}\frac{\epsilon}{18}\})^{\frac{1}{4}}. \end{aligned}$$

Since the random variables are nonnegatively correlated, we have

$$\begin{aligned} &\sigma^{-2}_{(n,n)} E(S_{([n(t_1+\delta)],[n(t_2+\delta)])} - S_{([nt_1],[nt_2])})^2 \\ &\le \sigma^{-2}_{(n,n)} (\sigma^2_{([n(t_1+\delta)],[n(t_2+\delta)])} - \sigma^2_{([nt_1],[nt_2])} \le 3\delta, \end{aligned}$$

according to (3.4). Hence by (3.5) it holds that

$$(3.6)\quad P\{w(W_n,\delta)>\epsilon\}\le 3^{\frac{3}{2}}2^{-\frac{1}{4}}\left(\frac{3\delta(18)^2}{\epsilon^2}\right)^{\frac{3}{4}}$$
$$\left(P\{|S_{([n(t_1+\delta)],[n(t_2+\delta)])}-S_{([nt_1],[nt_2])}|\ge\sigma_{(n,n)}\frac{\epsilon}{18}\}\right)^{\frac{1}{4}}.$$

According to (3.4), for $\epsilon_0<\frac{\epsilon}{18}$

$$P\{|S_{([n(t_1+\delta)],[n(t_2+\delta)])}-S_{([nt_1],[nt_2])}|\ge\sigma_{(n,n)}\frac{\epsilon}{18}\}$$
$$\le P\{\sigma^{-1}_{([n(t_1+\delta)]-[nt_1],[n(t_2+\delta)]-[nt_2])}\cdot$$
$$|S_{([n(t_1+\delta)],[n(t_2+\delta)])}-S_{([nt_1],[nt_2])}|\ge\epsilon_0\delta^{-1}\}$$

Then (3.6) yields

$$\lim_{n\to\infty}\sup P\{w(W_n,\delta)>\epsilon\}$$
$$\le 3^{\frac{3}{2}}2^{-\frac{1}{4}}\left(\frac{3\delta}{\epsilon_0^2}\right)^{\frac{3}{4}}\left(\frac{\delta}{\epsilon_0}\right)^2\Big[\sup_{m_1,m_2\in N\cup\{0\},n\in N}E\big(\sigma^{-2}_{(n,n)}(S_{(n+m_1,n+m_2)}-S_{(m_1,m_2)})^2\cdot$$
$$1_{\{|S_{(n+m_1,n+m_2)}-S_{(m_1,m_2)}|\ge\sigma_{(n,n)}\epsilon_0\delta^{-1}\}}\big)\Big]^{\frac{1}{4}}.$$

Now assumption (3.2) implies (3.3), and the proof of Theorem 3.1 is complete.

REMARK. For one parameter case, Theorem 3.1 was obtained by Birkel (1988).

THEOREM 3.2. *Let $\{X_{\underline{j}}:\underline{j}=(j_1,j_2),\ j_1,j_2\in N\}$ be a two-parameter array of associated random variables with $EX_{\underline{j}}=0$, $EX^2_{\underline{j}}<\infty$. If (3.1) and (3.2) are fulfilled, then $\{X_{\underline{j}}\}$ satisfies the invariance principle.*

Proof. Since $EW_n^2(\underline{0})=0$, $\{W_n(\underline{0})\}$ is certainly tight. It follows from (3.3) and Theorem 15.5 of Billingsley [2] that $\{W_n\}$ is tight. It remains to prove that the finite dimensional distributions of W_n converges to those of W. Let X be a limit in distribution of a subsequence of $\{W_n:n\in N\}$. Then $P\{X\in\mathcal{C}_2\}=1$ by Theorem 15.5 of [2]. By (3.2) and (3.4), $\{W_n^2(\underline{t}):n\in N\}$ and $\{W_n(\underline{t}):n\in N\}$ are uniformly integrable for every $\underline{t} \in [\underline{0}, \underline{1}]$. As $W_n(\underline{t}) \to X(\underline{t})$, $W_n^2(\underline{t}) \to X^2(\underline{t})$, in distribution (for a subsequence), Theorem 5.4 of [2] and (3.4) imply $EX(\underline{t}) = 0$, $EX^2(\underline{t}) = |\underline{t}|$. Since, for $\underline{0} \le \underline{t}_0 \le \underline{t}_1 \le \cdots \le \underline{t}_k \le \underline{1}$,

$$(3.7) \quad (W_n(\underline{t}_1) - W_n(\underline{t}_0), \ldots, W_n(\underline{t}_k) - W_n(\underline{t}_{k-1})) \\ \to (X(\underline{t}_1) - X(\underline{t}_0), \cdots, X(\underline{t}_k) - X(\underline{t}_{k-1}))$$

in distribution (for a subsequence), and the $W_n(\underline{t}_i) - W_n(\underline{t}_{i-1})$'s are associated by P_4 of Esary, Proschan and Walkup [7], $X(\underline{t}_1) - X(\underline{t}_0), \cdots, X(\underline{t}_k) - X(\underline{t}_{k-1})$ are associated, according to P_5 of [7]. According to Lemma 2.3, we have

$$(3.8) \quad \sigma_{(n,n)}^{-2} E((S_{[n\underline{t}]} - S_{[n\underline{s}]})(S_{[n\underline{v}]} - S_{[n\underline{u}]})) \to 0 \text{ for } \underline{0} \le \underline{s} \le \underline{t} \le \underline{u} \le \underline{v}.$$

Since Theorem 5.4 of Billingsley [2] and (3.8) yield that for $i \neq j$,

$$\mathrm{Cov}(X(\underline{t}_i) - X(\underline{t}_{i-1}), X(\underline{t}_j) - X(\underline{t}_{j-1})) \\ = \lim_{n\to\infty} \mathrm{Cov}(W_n(\underline{t}_i) - W_n(\underline{t}_{i-1}), W_n(\underline{t}_j) - W_n(\underline{t}_{j-1})) = 0.$$

Hence the $X(\underline{t}_i) - X(\underline{t}_{i-1})$ are associated and uncorrelated random variables and thus independent by Corollary 3 of Newman [11], i.e., X has independent increment. Therefore X is distributed like W according to Lemma 2 of Deo [6] and the proof of Theorem 3.2 is complete.

REMARK. If for Lemmas 2.2 and 2.3 can be extended to the d-parameter case ($d > 2$) then we can extend Theorem 3.1 (i.e., (3.3)) to the $d > 2$ case then like in the proof of Theorem 3.2, we prove an invariance principle for a d-parameter nonstationary associated processed ($d > 2$).

COROLLARY 3.3. *Let $\{X_{(j_1,j_2)} : j_1, j_2 \in N\}$ be a two-parameter array of associated random variables with $EX_{(j_1,j_2)} = 0$, $EX^2_{(j_1,j_2)} < \infty$. Assume*

$$(3.9) \quad n^{-2} E(S_{(nk_1,nk_2)} S_{(nl_1,nl_2)}) \to \sigma^2 k_1 k_2 \text{ for } k_i \le l_i \in N,\ i = 1, 2,$$

where $\sigma^2 \in (0, \infty)$,

$$(3.10) \quad \{n^{-2}(S_{(n+m_1,n+m_2)} - S_{(m_1,m_2)})^2 : m_1, m_2 \in N \cup \{0\},\ n \in N\}$$

is uniformly integrable. Then $\{X_{\underline{j}}\}$ fulfills the invariance principle.

Proof. Since $n^{-2}\sigma^2_{(n,n)} \to \sigma^2$, by putting $k_i = l_i = 1$ for $i = 1, 2$, (3.9) and (3.10) yield (3.1) and (3.2). Thus $\{X_{(j_1,j_2)}\}$ fulfills the invariance principle, according to Theorem 3.2.

The following theorem shows that the invariance principle still holds if condition (3.2) is replaced by the validity of the central limit theorem. Hence we obtain necessary and sufficient conditions for the invariance principle.

THEOREM 3.4. *Let $\{X_{(j_1,j_2)} : j_1, j_2 \in N\}$ be a two- parameter array of associated random variables with $EX_{(j_1,j_2)} = 0$, $EX^2_{(j_1,j_2)} < \infty$. Then the following assertion are equivalent:*

(i) *Condition* (3.1) *is fulfilled and $\{X_{(j_1,j_2)} : j_1, j_2 \in N\}$ satisfies the central limit theorem,*

(ii) *$\{X_{(j_1,j_2)} : j_1, j_2 \in N\}$ fulfills the invariance principle.*

Proof. It suffices to show that (i) $\Rightarrow$ (ii). Like in the proof of Theorems 3.1 and 3.2 we obtain relations (3.4) and (3.7). From the central limit theorem and (3.4) it follows that for $\underline{t} > \underline{0}$

$$\sigma^{-1}_{(n,n)} S_{[n\underline{t}]} \to N(0, |\underline{t}|) \quad \text{in distribution.} \tag{3.11}$$

Using the techniques in the proof of Theorem 2 of Birkel [3] we obtain for $\underline{0} < \underline{s} < \underline{t}$,

$$\sigma^{-1}_{(n,n)}(S_{[n\underline{t}]} - S_{[n\underline{s}]}) \to {}_nN(0, |\underline{t}| - |\underline{s}|) \quad \text{in distribution.} \tag{3.12}$$

The proof of Theorem 3.4 can now be completed along the lines of the proof of Theorems 3.1 and 3.2 (cf. the end of the proof of Theorem 2.2 of Herrendorf [9]): The crucial point is the verification of relation (3.3). (3.6) can be proved by the same argument, and then by an application of (3.12) one can obtain for $\epsilon_0 < \frac{\epsilon}{18}$

(3.13)
$$\begin{aligned} &P\{w(W_n, \delta) > \epsilon\} \\ &\le 3^{\frac{3}{2}} 2^{-\frac{1}{4}} \left(\frac{3\delta}{\epsilon_0^2}\right)^{\frac{3}{4}} \left(P\{|S_{([n(t_1+\delta)],[n(t_2+\delta)])} - S_{([nt_1],[nt_2])}| \ge \epsilon_0 \sigma_{(n,n)}\}\right)^{\frac{1}{4}} \\ &\le 3^{\frac{3}{2}} 2^{-\frac{1}{4}} \left(\frac{3\delta}{\epsilon_0^2}\right)^{\frac{3}{4}} \left(N(0, (t_1 + t_2 + \delta)\delta)\{x \in R : |x| \ge \epsilon_0\}\right)^{\frac{1}{4}}. \end{aligned}$$

Since for fixed ϵ_0, we have some constants B and b,

$$3^{\frac{3}{2}} \cdot 2^{-\frac{1}{4}}(\frac{3\delta}{\epsilon_0^2})^{\frac{3}{4}}(N(0,(t_1+t_2+\delta)\delta)\{x \in R : |x| \geq \epsilon_0\})^{\frac{1}{4}}$$
$$\leq B\delta^{\frac{3}{4}}(\int_{b\delta^{-1}}^{\infty}(2\pi)^{-\frac{1}{2}}\exp(-\frac{u^2}{2})du)^{\frac{1}{4}} \to 0 \text{ as } \delta \to 0,$$

(3.5), (3.6) and (3.13) yield (3.3) and completes the proof.

Theorem 3.4 shows that the central limit theorem is an important tool in establishing the invariance principle for associated processes.

COROLLARY 3.5. *Let* $\{X_{\underline{j}} : \underline{1} \leq \underline{j} \leq n\underline{1},\ n \in N\}$ *be an array of two parameter associated random variables with* $EX_{\underline{j}} = 0$, $EX_{\underline{j}}^2 < \infty$. *Assume that* (2.1), (2.2), (2.3) *and* (3.1) *are fulfilled. Then* $\{X_{\underline{j}} : \underline{1} \leq \underline{j} \leq n\underline{1},\ n \in N\}$ *fulfills the invariance principle.*

Proof. According to Theorem 2.1 it follows from (2.1), (2.2) and (2.3) that $\{X_{\underline{j}}\}$ satisfies the central limit theorem and hence Theorem 3.4 and (3.1) yield that $\{X_{\underline{j}}\}$ fulfills the invariance principle.

The following theorem shows that condition (3.1) is necessary for the invariance principle whereas condition (3.2) cannot be deduced from the invariance principle.

THEOREM 3.6. (i) *Let* $\{X_{\underline{j}} : \underline{j} = (j_1, j_2),\ j_1, j_2 \in N\}$ *be a two-parameter array of associated random variables with* $EX_{\underline{j}} = 0$, $EX_{\underline{j}}^2 < \infty$. *If* $\{X_{\underline{j}}\}$ *fulfills the invariance principle, then condition* (3.1) *holds.*

(ii) *There exists a two-parameter array* $\{X_{\underline{j}}\}$ *of associated random variables with* $EX_{\underline{j}} = 0$, $EX_{\underline{j}}^2 < \infty$, *which fulfills the invariance principle, but does not satisfy condition* (3.2).

Proof. (i). Assume that the invariance principle is fulfilled. Then for $\underline{t} = (t_1, t_2) \in (0,1]^2$, we have $\sigma^{-1}_{([nt_1],[nt_2])}S_{([nt_1],[nt_2])} \to N(0,1)$, $\sigma^{-1}_{(n,n)}S_{([nt_1],[nt_2])} \to N(0, t_1t_2)$ weakly as $n \to \infty$. Therefore $\sigma^{-2}_{(n,n)}\sigma^2_{([nt_1],[nt_2])} \to t_1t_2$, as $n \to \infty$, which yields, by Lemma 2.2

$$\sigma^{-2}_{(n,n)}\sigma^2_{(nk_1,nk_2)} \to k_1k_2 \quad \text{for} \quad k_1, k_2 \in N. \tag{3.14}$$

For $i = 1,2$ let $0 \le s_i \le t_i \le u_i \le v_i \le 1$ be given. Since the invariance principle is fulfilled, $\{\sigma_{(n,n)}^{-2} S_{(n,n)}^2 : n \in N\}$ is uniformly integrable. Hence $\{\sigma_{(n,n)}^{-2}(S_{([nt_1],[nt_2])} - S_{([ns_1],[ns_2])})(S_{([nv_1],[nv_2])} - S_{([nu_1],[nu_2])}) : n \in N\}$ is uniformly integrable, according to Lemma 2.2. As

$$\begin{aligned}\sigma_{(n,n)}^{-2}&((S_{([nt_1],[nt_2])} - S_{([ns_1],[ns_2])})(S_{([nv_1],[nv_2])} - S_{([nu_1],[nu_2])}))\\ &= (W_n(t_1,t_2) - W_n(s_1,s_2))(W_n(v_1,v_2) - W_n(u_1,u_2))\\ &\to (W(t_1,t_2) - W(s_1,s_2))(W(v_1,v_2) - W(u_1,u_2))\end{aligned}$$

it follows from Theorem 5.4 of Billingsley [2] and the fact that W has independent increment with mean zero,

(3.15)

$$\begin{aligned}\lim_{n\to\infty} \sigma_{(n,n)}^{-2} &E((S_{([nt_1],[nt_2])} - S_{([ns_1],[ns_2])})(S_{(nv_1],[nv_2])} - S_{([nu_1],[nu_2])}))\\ &= E((W(t_1,t_2) - W(s_1,s_2))(W(v_1,v_2) - W(u_1,u_2)))\\ &= E(W(t_1,t_2) - W(s_1,s_2))E(W(v_1,v_2) - W(u_1,u_2)) = 0.\end{aligned}$$

Hence Lemma 2.3, (3.14) and (3.15) yield that condition (3.1) holds.

(ii). Let $\{X_{(j_1,j_2)}\}$ be an array of independent random variables, such that the distribution of $X_{(j_1,j_2)}$ is $N(0,(j_1+1)\log(j_1+1) - j_1\log(j_1))$. According to Theorem 2.1 of Esary, Proschan and Walkup [7], the array $\{X_{(j_1,j_2)}\}$ is associated. Since the random variables are independent and normally distributed, the distribution of $S_{(n,m)}$ is $N(0, m(n+1)\log(n+1))$, and hence $\{X_{(j_1,j_2)} : j_1, j_2 \in N\}$ fulfills the central limit theorem. Moreover, for $k_i,\ l_i \in N$, $i = 1,2$,

$$\sigma_{(n,n)}^{-2} E(S_{(nk_1,nk_2)} S_{(nl_1,nl_2)}) \to \Pi_{i=1}^2 \min\{k_i, l_i\}.$$

According to Theorem 3.4, $\{X_{(j_1,j_2)} : j_1, j_2 \in N\}$ fulfills the invariance principle. But for $m_1', m_2' \in N$ there holds

$$\begin{aligned}\sup_{\substack{m_1,m_2 \in N\cup\{0\}\\ n\in N}} &\sigma_{(n,n)}^{-2} E(S_{(n+m_1,n+m_2)} - S_{(m_1,m_2)})^2\\ &\ge (2\log(2))^{-1} EX_{(m_1'+1,m_2'+1)}^2\\ &\ge (2\log(2))^{-1}\log(m_1'+2),\end{aligned}$$

and thus

$$\sup_{\substack{m_1,m_2\in N\cup\{0\}\\ n\in N}} \sigma^{-2}_{(n,n)} E(S_{(n+m_1,n+m_2)} - S_{(m_1,m_2)})^2 = \infty.$$

Hence $\{X_{(j_1,j_2)} : j_1, j_2 \in N\}$ does not satisfy condition (3.2).

THEOREM 3.7. *Let $\{X_{\underline{j}} : \underline{1} \leq \underline{j} \leq n\underline{1},\ n \in N\}$ be a family of two parameter associated random variables with $EX_{\underline{j}} = 0$, $EX^2_{\underline{j}} < \infty$. Assume that (2.1), (2.2) and (2.3) are fulfilled. If $\{X_{\underline{j}}\}$ satisfies*

$$(3.16) \qquad |\underline{n}|^{-1}\sigma^2_{\underline{n}} \to \sigma^2 \in (0,\infty),\ \underline{n} = (n_1, n_2),\ n_1, n_2 \in N,$$

then $\{X_{\underline{j}}\}$ fulfills the invariance principle.

Proof. According to Theorem 2.1, $\{X_{\underline{j}}\}$ satisfies the central limit theorem. It remains to prove (3.1). It follows from (3.16) that (i) of Lemma 2.2 is fulfilled. By a simple consequence of the estimate

$$0 \leq \sigma^{-2}_{(n,n)} E((S_{n\underline{j}} - S_{n\underline{i}})(S_{n\underline{l}} - S_{n\underline{k}})) \leq \sigma^{-2}_{(n,n)} \sum_{r=1}^{n|\underline{j}-\underline{i}|} u(r),$$

conditions (2.1) and (3.16) we have

$$(3.17) \quad \sigma^{-2}_{(n,n)} E((S_{n\underline{j}} - S_{n\underline{i}})(S_{n\underline{l}} - S_{n\underline{k}})) \to 0 \text{ for } \underline{0} \leq \underline{i} \leq \underline{j} \leq \underline{k} \leq \underline{l} \in Z^2.$$

Thus (3.1) is fulfilled according to Lemma 2.3.

REMARK. In order to prove an invariance principle for d-parameter associated processes it is necessary to extended Theorem 3.1 to the multiparameter case ($d \geq 3$). The question of whether the relation (3.3) holds for $d \geq 3$ is presently left open (See [13]).

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