Comm. Korean Math. Soc. **8** (1993), No. 4, pp. 741–754

GAUGE THEORY ON RAMIFIED COVERING SPACES

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§1. Introduction

Let X be a smooth oriented 4-manifold. Suppose that a cyclic group $\mathbb{Z}_n$ acts smoothly on X. Let $P : E \to X$ be an $SU(2)$-vector bundle over X with a smooth G-action such that the projection P is a G-map. Let $\pi : X \to X' = X/\mathbb{Z}_n$ be the projection and $E \to E' = E/\mathbb{Z}_n$ be the quotient bundle of E.

In this paper we would like to study the smooth structures on X', the relation between the moduli space of $\mathbb{Z}_n$-invariant anti-self-dual connections on E and the moduli space of anti-self-dual connections on E', and the relation between the polynomial invariants which is defined regarding the invariant moduli space $\mathcal{M}^{\mathbb{Z}_n}$ and the polynomial invariants which is defined by the moduli space $\mathcal{M}'$ on the quotient bundle E'.

In [F.S] and [C1] they showed that there exists a Baire set in the G-invariant metrics on X, when the manifold X has a finite group G-action, such that the moduli space $\mathcal{M}^G$ of G-invariant self-dual connections is smooth except the reducible singularities. In [C1] by using the G-transversality argument of T. Petrie, we identify cohomology obstructions to globally perturb the full moduli space $\mathcal{M}$ of all self-dual connections into a G-manifold when $G = \mathbb{Z}_2$ and the fixed point set of the G-action on X is a non-empty collection of isolated points and Riemann surfaces. In [C2] we find generic metrics on X such that the moduli space $\mathcal{M}$ is smooth in a G-invariant neighborhood of the fixed point set $\mathcal{M}^G$ when $G = \mathbb{Z}_{2^n}$, for a Baire set of invariant metrics on X. In [H.L] they show that when G is a finite group, the G-equivariant moduli space $\mathcal{M}^*$ has a Whitney stratification with invariant subspaces $\mathcal{M}^*_{G'}$, $G' \subseteq G$ as its strata, by perturbing the self-dual equations and Bierstone's general position argument of equivariant maps in finite dimensional manifolds.

Received May 9, 1993.
The first author was supported by GARC-KOSEF.

In [W] when the group G is $\mathbb{Z}_2$ he gives the relations between the invariant moduli space $\mathcal{M}^G$ on a G-bundle $E \to X$ and the moduli space $\mathcal{M}'$ on the quotient bundle $E' \to X'$ and gives a relation between polynomial invariants defined by them. In [C4] he find generic G-invariant metric on X such that the moduli space $\mathcal{M}$ is a smooth G-manifold except the reducible singularities if the instanton number $c_2(E)$ is large enough.

§2. Lipschitz structure

A topological manifold X of dimension 4 is *Lipschitz* if there is a maximal atlas $\{U_\alpha, \phi_\alpha\}_{\alpha\in\Lambda}$ on X, where $\phi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^4$ is a homeomorphism from an open set $U_\alpha \subset M$ onto an open set V_α of $\mathbb{R}^4$, and the changes of coordinates $\phi_\beta \circ \phi_\alpha^{-1}$ are Lipschitz functions, i.e., $|\phi_\beta\phi_\alpha^{-1}(x) - \phi_\beta\phi_\alpha^{-1}(y)| \leq K_{\alpha\beta}|x-y|$ for any $x, y \in \phi_\alpha(U_\alpha \cap U_\beta)$ with $K_{\alpha\beta}$ a constant.

In [S] Sullivan defined L_2-forms, exterior derivatives and differential forms on the Lipschitz manifolds. An L_2-form w of degree r on X is a system, $w = \{w_\alpha\}_{\alpha\in\Lambda}$, where each w_α is a L_2-differential form of degree r on the open subset $V_\alpha = \phi_\alpha(U_\alpha)$ of $\mathbb{R}^4$, and they satisfy the compatibility conditions:

$$(\phi_\beta\phi_\alpha^{-1})^* w_\beta = w_\alpha.$$

PROPOSITION 2.1(RADEMACHER). *Let U be an open subset of $\mathbb{R}^4$, and let $\varphi : U \to \mathbb{R}^4$ be a Lipschitz map, then;*

(i) *φ is differentiable almost everywhere on U*

(ii) *$\nabla\varphi$ is a weak derivative :*

$$\int_U f\frac{\partial\varphi}{\partial x_i} = -\int \frac{\partial f}{\partial x_i}\varphi$$

for smooth compactly supported test functions f on U.

(iii) *φ preserves Lebesgue null sets.*

THEOREM 2.2 [S]. *Any topological manifold of dimension $\neq 4$ has a Lipschitz atlas of coordinates, and for any two such Lipschitz structures $\mathcal{L}_i$, $i = 1, 2$, there exits a Lipschitz homeomorphim $h : \mathcal{L}_1 \to \mathcal{L}_2$ close to the identity.*

THEOREM 2.3 [D.S]. (i) *There are topological* 4*-manifolds which do not admit any Lipschitz structure.*

(ii) *There are Lipschitz* 4*-manifolds which are homeomorphic but not Lipschitz equivalent.*

In [D.S] Donaldson and Sullivan studied the gauge theory on the quasiconformal 4-manifolds. As consequences they showed that the compact simply connected topological 4-manifolds with negative definite, even intersection forms do not admit quasiconformal structure. They showed that the complex Barlow surface is not quasiconformally equivalent to $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$ (they are homeomorphic). Similarly we may establish the gauge theory on the Lipschitz 4-manifolds. We may apply their results to the Lipschitz 4-manifolds. Then we will get Theorem [D.S].

§3. Smooth structure on quotient spaces

Let X be a smooth oriented, closed 4-manifold. Suppose that the cyclic group $\mathbb{Z}/n$ of order n acts semifreely on X with a 2-dimensional submanifold B as its fixed point set, and let $X' = X/\mathbb{Z}_n$ be its quotient space. Then we have an n-fold ramified covering space:

$$\pi : X \to X'$$

with branching locus $\pi(B) = B'$.

To study the smooth structures on X' we consider a small tubular neighborhood N of B in X which is isomorphic to the normal bundle of B in X. The projection π gives rise to a tubular neighborhood $\pi(N) = N'$ of $\pi(B) = B'$ in X' which is also isomorphic to the normal bundle of B' in X'. For a coordinate system $\{B_\alpha\}$ of B, let $N_\alpha \to B_\alpha \times \mathbb{C}$ be a local trivialization of the normal bundle $N \to B$, given by $(b, v) \mapsto (b, \varphi_\alpha(v))$.

THEOREM 3.1. *If we give a local trivialization* $\pi(N_\alpha) = N'_\alpha \to \pi(B_j) \times\mathbb{C} = B'_\alpha \times \mathbb{C}$ *on the normal bundle* $N' \to B'$ *by* $(\pi(b), \pi(v)) \mapsto (\pi(b), \varphi_\alpha(v)^n)$, *then*

(i) *the quotient space* X' *is a smooth* 4*-manifold,*
(ii) *the projection map* $\pi : X \to X'$ *is smooth,*
(iii) *but* π *is not Lipschitz.*

Proof. (i) Suppose that the fixed point set B is orientable. Then the normal bundle $N \to B$ is a $U(1)$-bundle. The transition map $\varphi_\beta\varphi_\alpha^{-1}$: $B_\alpha \cap B_\beta \to U(1)$ gives an attaching linear map $(b,v) \mapsto (b, e^{i\theta}v)$, where $e^{i\theta} = \varphi_\beta\varphi_\alpha^{-1}(b)$ on the normal bundle N. By definition this transition map induces the transition map $\varphi'_\beta\varphi'^{-1}_\alpha : B'_\alpha \cap B'_\beta \to U(1)$ given by $(b',v') \mapsto (b', e^{in\theta}v')$, i.e., $\varphi'_\beta\varphi'^{-1}_\alpha(b') = e^{in\theta}$, and $|v|^n = |v'|$ if $\pi(b,v) = (b',v')$. Thus $N' \to B'$ is a smooth $U(1)$-bundle.

If B is not orientable, then we consider the orientation line bundle $L \to B$ which is given by transition functions, the Jacobian determinant of the matrix of partial derivatives of the transition functions of the tangent bundle TB. We can get the same result by tensoring the orientation bundle L to the normal bundle N.

(ii) On the free part the quotient space X' has the smooth structure induced by the smooth structure of X. Near the fixed point B the projection map π is just the bundle map between the normal bundles N and N'. Thus the projection map π is smooth.

(iii) Near the fixed point set, $\pi : N \to N'$ is the normal bundle map. We consider the local trivializations as above (i)

$$\begin{array}{ccccc} B_\alpha & \longleftarrow & N|_{B_\alpha} & \xrightarrow{\cong} & B_\alpha \times \mathbb{C} \\ {\scriptstyle \pi = id}\downarrow & & {\scriptstyle \pi}\downarrow & & {\scriptstyle id}\downarrow \\ B'_\alpha & \longleftarrow & N'|_{B'_\alpha} & \xrightarrow{\cong} & B'_\alpha \times \mathbb{C} \end{array}$$

since π = identity on the fixed point set, $\pi(b,z) = (b, z^n)$ and $|(b,0) - (b,z)| = |z|$, $|\pi(b,0) - \pi(b,z)| = |z|^n$. Thus π is not Lipschitz.

THEOREM 3.2. *If we give a local trivialization $N'_\alpha \equiv \pi(N_\alpha) \to \pi(B_\alpha) \times \mathbb{C}$ on the normal bundle $N' \to B'$ in X' by*

$$(\pi(b), \pi(v)) \mapsto (\pi(b), \frac{\varphi_\alpha(v)^n}{|\varphi_\alpha(v)|^{n-1}}),$$

then (i.e., in the polar coordinate $(\pi(b), \pi(v)) \mapsto (\pi(b), re^{in\theta})$ if $\varphi_\alpha(v) = re^{i\theta}$)

(i) *X' is smooth,*
(ii) *π is not smooth,*
(iii) *but π is bi-Lipschitz.*

Proof. (i) On the free part the Theorem is obvious because the structure on X' comes from the structure on X. It is enough to prove the theorem on a small tubular neighborhood of the fixed point set B. The proof is similar to the proof of above theorem. As a normal bundle, if $N \to B$ has the transition function $e^{i\theta} \in U(1)$, then the transition function of $N' \to B'$ is $e^{in\theta} \in U(1)$. Thus the X' is smooth.

(ii) In the local trivialization of the normal bundle, the projection $\pi(b, re^{i\theta}) = (\pi(b), re^{in\theta})$ is not smooth because $re^{in\theta} = \frac{\varphi_\alpha(v)^n}{|\varphi_\alpha(v)|^{n-1}}$ is not smooth when $\varphi_\alpha(v) = 0$, where $\varphi_\alpha(v) = re^{i\theta}$. Thus π is not smooth on the fixed point set B.

(iii) Let $\varphi(v_1) = r_1 e^{i\theta_1}$, $\varphi(v_2) = r_2 e^{i\theta_2}$. Then

$$|\varphi(v_1) - \varphi(v_2)| = (r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1))^{\frac{1}{2}} \quad \text{and}$$
$$\left|\frac{\varphi(v_1)^n}{|\varphi(v_1)|^{n-1}} - \frac{\varphi(v_2)^n}{|\varphi(v_2)|^{n-1}}\right| = (r_1^2 + r_2^2 - 2r_1r_2\cos n(\theta_2 - \theta_1))^{\frac{1}{2}}.$$

There are constants c_1 and c_2 so that

$$c_1|\varphi(v_1) - \varphi(v_2)| \le \left|\frac{\varphi(v_1)^n}{|\varphi(v_1)|^{n-1}} - \frac{\varphi(v_2)^n}{|\varphi(v_2)|^{n-1}}\right| \le c_2|\varphi(v_1) - \varphi(v_2)|.$$

In this paper we will assume that X' has the smooth structure of Theorem 3.2. Since π is a bi-Lipschitz map, π sends sets of measure zero into sets of measure zero, moreover π sends measurable sets into measurable sets and so is π^{-1}.

From Theorem 3.2 we have following corollary.

COROLLARY 3.3. (i) *The $\mathbb{Z}/n$-invariant smooth metrics or differential forms on X pushdown to the metrics or forms on X' which are smooth away from B', and have bounded coefficients near B'.*

(ii) *The projection $\pi : X \to X'$ induces an 1-1 correspondence between $\mathbb{Z}/n$-invariant L^p-metrics (forms) on X and L^p-metrics (forms) on X'.*

Proof. (i) Since the smoothness is a local property and the restriction $\pi : X\backslash B \to X'\backslash B'$ is locally diffeomorphic, it is sufficient to prove the theorem near B'. At near B the metric is of the form $dx_1^2 + dx_2^2 + dr^2 + d\theta^2$, the pushdown metric is of the form $dx_1'^2 + dx_2'^2 + dr'^2 + d\theta'^2 = dx_1^2 + dx_2^2 + dr^2 + nd\theta^2$ since $\pi(x_1, x_2, r, \theta) = (x_1', x_2', r', \theta') = (x_1, x_2, r, n\theta)$. In differential forms since $\theta' = n\theta$, they have bounded coefficients near B'.

(ii) If we push down the $\mathbb{Z}_n$-invariant L^p-metrics (forms) on X then they are the L^p-metrics (forms) on X'. Conversely the pullback of the L^p-metric (form) on X' is a $\mathbb{Z}_n$-invariant L^p-metric (form) on X.

We consider some topological consequences on the quotient space X'. If X is simply connected, then $\pi_* : \pi_1(X) \to \pi_1(X')$ is zero. Indeed if we choose a base point in the branching locus B', any loop at the base point can meet only the point with B' by small perturbation. That loop lifts n-distinct loops in X with a base point in B. Thus X' is also simply connected. In [C3] to prove the following proposition we use the Hirzebruch G-signature Theorem.

THEOREM 3.4 [C3].

(1) *The signature of X' : $sign(X') = \frac{1}{n}(sign(X) + \frac{n^2-1}{2} B \circ B)$*
(2) *The Euler characteristic of X' : $\chi(X') = \frac{1}{n}(\chi(X) + (n-1)\chi(B))$*
(3) *The rank of the maximal subspace of $H^2(X)$ which consists of the self-dual harmonic 2-forms:*

$$b_2^+(X') = \frac{1}{n}\left[b_2^+(X) + \frac{n-1}{2}\{\chi(B) + \frac{n+1}{3} B \circ B - 2\}\right]$$

where $B \circ B$ is the self-intersection number of B in X.

§4. Comparison between Equivariance and Quotient

Let $E' \to X'$ be an $SU(2)$-vector bundle with the second Chern number $c_2(E') = k'$ on the quotient X'. Let the pull-back bundle $E = \pi^* E'$ and $\mathbb{Z}/n = \langle\sigma\rangle$ be generated by σ and its lifting $\tilde{\sigma}$ on E. Then the second Chern number of E, $c_2(E) = nk' = k$. The lifting $\tilde{\sigma}$ is the identity on B, and is smooth away from B and Lipschitz around B as a bundle map $\tilde{\sigma} : E \to E$.

For $p > 4$, we would like to define a modified Sobolev space of 1-forms on X;

$$\tilde{L}^p(\Omega_X^1) = \{\alpha \in L^p(\Omega_X^1) \,|\, d\alpha \in L^{p/2}(\Omega_X^2)\},$$

$\tilde{\mathcal{A}}^p = \{A_0 + a \,|\, a \in \tilde{L}^p(\Omega_X^1(ad\, E))\}$, where A_0 is smooth, $\tilde{\mathcal{A}}^{p,\sigma}$ is the $\mathbb{Z}/n$-invariant subspace of $\tilde{\mathcal{A}}^p$.

PROPOSITION 4.1. *(i) The projection map $\pi : E \to E'$ induces a bijection $\tilde{\mathcal{A}}^{p,\sigma} \to \tilde{\mathcal{A}}'^{p}$ where $\tilde{\mathcal{A}}'$ is the modified Sobolev space of connections on E'.*

(ii) For an anti-self-dual, σ-invariant connection $A \in \tilde{\mathcal{A}}^{p,\sigma}$ the push down connection A' on X' is also anti-self-dual.

(iii) The diagram

$$\begin{array}{ccccc} L_1^p(ad\,E)^\sigma & \xrightarrow{d_A} & \hat{L}^p(\Omega_X^1(ad\,E))^\sigma & \xrightarrow{d_A^+} & L_+^{p/2}(\Omega_X^2(ad\,E))^\sigma \\ \downarrow & & \downarrow & & \downarrow \\ L_1^p(ad\,E') & \xrightarrow{d_{A'}} & \tilde{L}^p(\Omega_{X'}^1(ad\,E')) & \xrightarrow{d_{A'}^+} & L_+^{p/2}(\Omega_{X'}^2(ad\,E')) \end{array}$$

is commutative for an anti-self-dual connection A in $\tilde{\mathcal{A}}^{p,\sigma}$. The above two complexes are isomorphic and elliptic.

In [D.S] to compute the index of the above elliptic complex we may use the parametrix $Q_{A'}$ for $d_{A'}^+$ instead of the adjoint operator $d_{A'}^* = -*d_{A'}*$ since the push down metric on X' of a G-invariant metric on X is singular. If we use the excision principle of the Atiyah-Singer Index Theorem, we have the index.

THEOREM 4.2. *The index of the above elliptic complex is*

$$\begin{aligned} i_{E'} &= 8c_2(E') - 3[1 - b_1(X') + b_2^+(X')] \\ &= \frac{1}{n}[i_E - \frac{3(n-1)}{2}\{\chi(B) + \frac{n+1}{3} B \circ B\}] \\ &= i_E^\sigma . \end{aligned}$$

In [D.S] Donaldson and Sullivan use a similar method to compute the index for the quasi-conformal setting.

§5. Equivariant generic metric

We would like to introduce equivariant generic metric on a 4-manifold X to define Donaldson polynomial invariant by regarding the equivariant moduli space $\mathcal{M}^{\mathbb{Z}/n} \subset \mathcal{B}^{\mathbb{Z}/n}$ as carrying a distinguished invariant homology class, independent of the choice of metric used to define $\mathcal{M}^{\mathbb{Z}/n}$.

Let $E \xrightarrow{p} X$ be an $SU(2)$-vector bundle with $c_2 = nk' = k$ over a simply connected closed smooth 4-manifold X. Suppose a cyclic group $\mathbb{Z}/n$ acts on X with a 2-dimensional submanifold B as the fixed point set. Let $\mathbb{Z}/n$ lift to the bundle E such that the projection p is a $\mathbb{Z}/n$-map. Choose $\mathbb{Z}/n$-invariant metrics on X and E. Recall that the notations $\mathcal{A}(\mathcal{A}^{\mathbb{Z}/n})$, $\mathcal{G}(\mathcal{G}^{\mathbb{Z}/n})$ the space of ($\mathbb{Z}/n$-invariant) connections and the group of ($\mathbb{Z}/n$-invariant) gauge transformations on E respectively.

Consider the fundamental elliptic complexes

$$0 \to \Omega^0(ad\,E) \to \Omega^1(ad\,E) \to \Omega^2_+(ad\,E) \to 0$$

and

$$0 \to \Omega^0(ad\,E)^{\mathbb{Z}/n} \to \Omega^1(ad\,E)^{\mathbb{Z}/n} \to \Omega^2_+(ad\,E)^{\mathbb{Z}/n} \to 0$$

Then the space $\mathcal{A}$ and $\mathcal{A}^{\mathbb{Z}/n}$ are affine spaces modeled on the spaces $\Omega^1(ad\,E)$ and $\Omega^1(ad\,E)^{\mathbb{Z}/n}$, and the groups of gauge transformations $\mathcal{G}(E)$ and $\mathcal{G}(E)^{\mathbb{Z}/n}$ are modeled on the spaces $\Omega^0(ad\,E)$ and $\Omega^0(ad\,E)^{\mathbb{Z}/n}$ respectively. Since the gauge transformations act on the connection spaces, we have the orbit spaces

$$\mathcal{B} = \mathcal{A}/\mathcal{G}(E) \quad \text{and} \quad \mathcal{B}^{\mathbb{Z}/n} = \mathcal{A}^{\mathbb{Z}/n}/\mathcal{G}(E)^{\mathbb{Z}/n}$$

We have some immediate consequences.

PROPOSITION 5.1. *Let $E' \xrightarrow{p'} X'$ be the quotient bundle of $E \xrightarrow{p} X$ under the $\mathbb{Z}/n$-action.*

(i) *X' has a smooth structure such that π is smooth.*
(ii) *$c_2(E) = nc_2(E')$.*
(iii) *The natural map $\mathcal{B}^{\mathbb{Z}/n} \to \mathcal{B}$ is injective if we restrict to irreducible connections and if the center of gauge group is trivial.*
(iv) *If we choose the pullback metric g on X from a smooth metric g' on X', then the metric g is $\mathbb{Z}/n$-invariant.*

Proof. (i) Let $\pi : X \to X'$ be the projection map. If N is a tubular neighborhood of B, then $N \to B$ is an $U(1)$-bundle and $N' = \pi(N) \to B' = \pi(B)$ is also an $U(1)$-bundle. If we identify B and B', then N' is isomorphic to $N \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} N$ of n-copies of N

(ii) Since $E = \pi^*(E')$, $p_1(E) = p_1\pi^*E' = \pi^*p_1(E') = np_1(E')$ and $c_2(E) = nc_2(E')$.

(iii) Suppose $A_1, A_2 \in B^{*\mathbb{Z}/n}$ and $A_1 = gA_2g^{-1}$ for some $g \in \mathcal{G}(E)$. Then $A_1 = h^{-1}ghA_2h^{-1}g^{-1}h = gA_2g^{-1}$, for any $h \in \mathbb{Z}/n$ and $g^{-1}h^{-1}ghA_2hg^{-1}hg = A_2$. $g^{-1}h^{-1}gh$ is an element of the center of the gauge group. Since the center is trivial, $gh = hg$ for any $h \in \mathbb{Z}/n$ and $g \in \mathcal{G}(E)^{\mathbb{Z}/n}$.

(iv) Since $\pi^*g' = g$, for any $v, w \in T_pX$, $g_p(v,w) = (\pi^*g')_p(v,w) = g'_{\pi(p)}(\pi_*v, \pi_*w)$ and for any $h \in \mathbb{Z}/n$, $gh_{(p)}(h_*v, h_*w) = (\pi^*g')_{h(p)}(h_*v, h_*w) = g'_{\pi(h(p))}(\pi_*h_*v, \pi_*h_*w) = g'_{\pi(p)}(h_*v, h_*w)$.

Let $\mathcal{U} = \mathcal{U}(GL(TX))$ be the set of c^k-automorphisms of the tangent bundle for a sufficiently large k. Let $\mathcal{U}^{\mathbb{Z}/n}$ be the subspace of $\mathbb{Z}/n$-invariant metrics in $\mathcal{U}$.

In fact if g is a fixed $\mathbb{Z}/n$-invariant metric on X, then every $\mathbb{Z}/n$-invariant metric on X is realized by a pull-back metric $\phi^*(g)$ of g for some $\phi \in \mathcal{U}^{\mathbb{Z}/n}$.

Let $p_+ : \Omega^{2,\mathbb{Z}/n} \to \Omega_+^{2,\mathbb{Z}/n}$ be the projection onto the self-dual $\mathbb{Z}/n$-invariant 2-forms with respect to the $\mathbb{Z}/n$-invariant metric g. Then $\phi^*p_+\phi^{-1*}$ is the projection onto self-dual, $\mathbb{Z}/n$-invariant 2-forms with respect to the metric $\phi^*(g)$.

We define a map $\Phi : B^{*\mathbb{Z}/n} \times \mathcal{U}^{\mathbb{Z}/n} \to \Omega^{2+}(ad\, E)^{\mathbb{Z}/n}$ by $\Phi(A, \phi) = p_+\phi^{-1*}F_A$. The map Φ is well-defined. Clearly a connection ∇ is anti-self-dual if and only if $h(\nabla)$ is anti-self-dual with respect to the metric $(h\phi)^*(g)$ [C3].

THEOREM 5.2 [F.U], [C1]. *The map Φ is smooth and has zero as a regular value. The inverse image $\Phi^{-1}(0)$ is an infinite dimensional Banach manifold of anti-self-dual connections parametrized by the space $\mathcal{U}^{\mathbb{Z}/n}$ of all $\mathbb{Z}/n$-invariant metrics on X.*

Consider the projection map

$$\phi : \Phi^{-1}(0) = \cup_{\phi \in \mathcal{U}^{\mathbb{Z}/n}} \mathcal{M}^{*\mathbb{Z}/n}_{\phi^*(g)} \to \mathcal{U}^{\mathbb{Z}/n}$$

which is a Fredholm map with $\mathbb{Z}/n$-invariant index of the fundamental $\mathbb{Z}/n$-invariant elliptic complex as its index. By the Sard-Smale Theorem for a Fredholm map between paracompact Banach manifolds, we have the following theorem.

THEOREM 5.3 [C1]. *There is a Baire set $\mathcal{U}'$ of $\mathcal{U}^{\mathbb{Z}/n}$ such that $\pi^{-1}(\phi) = \mathcal{M}^{\mathbb{Z}/n}_{\phi^*(g)}$ is a smooth manifold in the moduli space $\mathcal{M}_{\phi^*(g)}$ of the irreducible anti-self-dual connections for each metric $\phi \in \mathcal{U}'$.*

Let $g_0, g_1 \in \mathcal{U}'$, and $\gamma : [0,1] \to \mathcal{U}^{\mathbb{Z}/n}$ be a path between them. If $b_2^+(X') > 1$, by the similar proof as the above proposition we can perturb the path γ to a new path γ' which lies in $\mathcal{U}'$, that is, transverse to π. We may assume $\gamma(0) = \gamma'(0) = g_0$ and $\gamma(1) = \gamma'(1) = g_1$. Then we have a cobordism W_γ between the moduli spaces $\mathcal{M}^*(g_0)^{\mathbb{Z}/n}$ and $\mathcal{M}^*(g_1)^{\mathbb{Z}/n}$. For any ℓ we can choose the path γ' to have the transverse condition for all bundle E with $c_2(E) \le \ell$. Also if $b_2^+(X') > 1$ we may choose the path γ' to lie in $\mathcal{U}'$. Let us consider the reducible connection of E. If ∇ is a $\mathbb{Z}/n$-invariant reducible connection, then there is an equivariant bundle decomposition $E = L \oplus L^{-1}$ and an equivariant decomposition $\nabla = \nabla_0 \oplus \overline{\nabla}_0$ of the connection ∇. In addition if ∇ is anti-self-dual, then the curvature form $(i/2\pi)F$ represents the Euler class of the line bundle L and is a $\mathbb{Z}/n$-invariant ASD harmonic 2-form on X. The $\mathbb{Z}/n$-invariant fundamental elliptic complex

$$0 \to \Omega^0(ad\,E)^{\mathbb{Z}/n} \to \Omega^1(ad\,E)^{\mathbb{Z}/n} \to \Omega^2_+(ad\,E)^{\mathbb{Z}/n} \to 0$$

reduces to

$$(*) \qquad 0 \to \Omega^{0,\mathbb{Z}/n} \to \Omega^{1,\mathbb{Z}/n} \to \Omega^2_+(da\,E)^{\mathbb{Z}/n} \to 0$$

since the adjoint bundle is trivial. The index of $(*)$ is $(1/2)(\chi^{\mathbb{Z}/n} + \sigma^{\mathbb{Z}/n}) = 1 + b_2^{+.\mathbb{Z}/n}$, since $\dim H^{0,\mathbb{Z}/n} = 1$ and $\dim H^{1,\mathbb{Z}/n} - \dim H^{2,\mathbb{Z}/n} = -b_2^{+.\mathbb{Z}/n}$.

If $b_2^{+.\mathbb{Z}/n} > 0$ the Sard-Smale Theorem induces that generically there are no anti-self-dual solutions. Since the dimension of $\mathbb{Z}/n$-invariant, self-dual harmonic 2-forms on X equals to the dimension of self-dual harmonic 2-forms on the quotient space X' :

$$\begin{aligned} &b_2^{+\mathbb{Z}/n}(X) \\ &= b_2^+(X') \\ &= (1/n)[b_2^+(X) + ((n-1)/2)\{\chi(B) + ((n+1)/3)B \circ B - 2\}]. \end{aligned}$$

Thus we have the following proposition.

PROPOSITION 5.4. (1) *If* $b_2^+(X) > ((n-1)/2)\{2-\chi(B)-((n+1)/3)B \circ B\}$, *then there is an open dense subset* $\mathcal{U}'$ *of the* $\mathbb{Z}/n$*-invariant metrics* $\mathcal{U}^{\mathbb{Z}/n}$ *on* X *such that* $\mathcal{M}_g^{\mathbb{Z}/n}$ *does not have a reducible* $\mathbb{Z}/n$*-invariant ASD-connection for each* $g \in \mathcal{U}'$.

(2) *If* $b_2^+(X) > n + (n-1)/n\{2-\chi(B)-((n+1)/3B \circ B\}$, *then any path in* $\mathcal{U}^{\mathbb{Z}/n}$ *of metrics in* $\mathbb{Z}/n$*-invariant metrics joining two metrics* g_1 *and* g_2 *in* $\mathcal{U}'$ *can be perturbed into a new path in* $\mathcal{U}'$.

(3) *The moduli spaces* $\mathcal{M}^{\mathbb{Z}/n}$ *of the equivariant classes of* $\mathbb{Z}/n$*-invariant ASD-connections are smooth manifolds under the condition* (1), *are cobordant under the condition* (2) *for the invariant generic metrics in* $\mathcal{U}'$.

§6. Polynomial invariants

We need the following conditions to compute Donaldson polynomial invariant for the quotient bundle $E' \to X'$ (see [D.K]).

(i) The dimension of the moduli space, $\dim \mathcal{M}_{E'} = i_{E'} = 2d'$ is even.

(ii) To avoid nontrivial reducible connections and to get generic metric

$$b_2^+(X) > n - \frac{n-1}{2}[\chi(B) + \frac{n+1}{3}B \circ B - 2].$$

(iii) The stable range

$$4k' \geq 2 + 3[1 + \frac{1}{n}b_2^+(X) + \frac{n-1}{2n}\{\chi(B) + \frac{n+1}{3}B \circ B - 2\}].$$

To define the Donaldson invariant for the invariant setting we would like to introduce the Donaldson μ-map.

For $\Sigma \in H_2(X;\mathbb{Z})^{\mathbb{Z}/n}$ with $\pi_*\Sigma = n[\Sigma/\mathbb{Z}_n] \in H_2(X';\mathbb{Z})$, if A is $\mathbb{Z}/n$-invariant connection in $\tilde{\mathcal{A}}^p$, we have a coupled Dirac operator

$$\not{\partial}_A : \Gamma(S^+ \otimes E)^\sigma \to \Gamma(S^- \otimes E)^\sigma$$

where $S^\pm$ is the $(\pm\frac{1}{2})$-spinor bundle on Σ.

We have the determinant line bundle

$$\begin{array}{ccc} \gamma^*(\mathcal{L}_\Sigma) & & \mathcal{L}_\Sigma \\ \downarrow & & \downarrow \\ \mathcal{B}(X)^{\mathbb{Z}/n} & \xrightarrow{\ \gamma\ } & \mathcal{B}(\Sigma) \end{array}$$

with the fiber $(\det \operatorname{ind} \not{\partial}_A)^{-1}$, where γ is the restriction map.

Note that we should use a small tubular neighborhood $N(\Sigma)$ of Σ instead of Σ. There is a generic section s_Σ of this bundle such that

$$s_\Sigma^{-1}(0) = V_\Sigma \text{ is a codimension 2-submanifold of } \mathcal{M}_k^\sigma \subset \mathcal{B}_k^\sigma.$$

V_Σ is the Poincaré dual of the first Chern class $c_1(\mathcal{L}_\Sigma)$ of the determinant line bundle $\mathcal{L}_\Sigma$. This is the image $\mu(\Sigma)$ of the Donaldson map $\mu : H_2(X;\mathbb{Z})^\sigma \to H^2(\mathcal{B}^\sigma;\mathbb{Z})$. Define the Donaldson invariant for the invariant setting

$$q^\sigma : \operatorname{sym}^{d'}(H_2(X;\mathbb{Z})) \to \mathbb{Z} \quad \text{by}$$
$$q^\sigma(\Sigma_1, \ldots, \Sigma_{d'}) = \#\mathcal{M}_k^\sigma \cap V_{\Sigma_1} \cap, \ldots, \cap V_{\Sigma_{d'}}.$$

Here the number is counted with sign, and $\mathcal{M}_k^\sigma \cap V_{\Sigma_1}, \ldots, V_{\Sigma_{d'}}$ is a zero-dimensional compact manifold, hence finite.

In [C1] and [F.S] they showed that for the generic G-invariant metrics on X, the moduli space $\mathcal{M}_k^\sigma$ of G-invariant anti-self-dual connections is a smooth manifold.

PROPOSITION 6.1 [C3]. *The image of the first Chern class is*

$$\pi^*(c_1(\mathcal{L}_\Sigma)) = n.c_1(\mathcal{L}_{\Sigma/\mathbb{Z}_n}), \text{ and}$$

$$\begin{array}{ccc} H^2(X,\mathbb{Z})^\sigma & \xleftarrow[\pi^*]{} & H^2(X' : \mathbb{Z}) \\ PD\downarrow & & \downarrow n.PD \\ H_2(X;\mathbb{Z})^\sigma & \xrightarrow{\ \pi_*\ } & H_2(X' : \mathbb{Z}) \end{array}$$

is commutative where PD stands for Poincaré dual.

THEOREM 6.2. *If* $\pi_* : H_2(X;\mathbb{Z})^{\mathbb{Z}/n} \to H_2(X';\mathbb{Z})$ *and* $\pi_*(\xi_i) = n.\eta_i$, $i = 1, \ldots, d'$ *then* $q^\sigma(\xi_i, \ldots, \xi_{d'}) = q'(\eta_1, \ldots, \eta_{d'})$.

Sketch of the proof. Take a smooth generic metric g' on X'. The pull-back metric $g = \pi^* g'$ is a bounded measurable σ-invariant metric on X. Let $\{g_n\}$ be a sequence of σ-invariant generic smooth metric on X such that $g_n \to g$ in C^r-sense. Then the push-down metric g'_n of g_n on X' is bounded measurable and g'_n converges to g'. So far we showed that

(i) $q^\sigma_n(\xi_1, \ldots, \xi_{d'}) = \tilde{q}'_n(\eta_1, \ldots, \eta_{d'})$ for the metric g_n and g'_n respectively, where $\tilde{q}'_n$ is the polynomial invariant with respect to g'_n.

(ii) For large n

$$\tilde{q}'_n(\eta_1, \ldots, \eta_{d'}) = q'(\eta_1, \ldots, \eta_{d'}) \text{ (see [D.S] for detail).}$$

Combining (i) and (ii), we have the required result,

$$q^\sigma(\xi_1, \ldots, \xi_{d'}) = q'(\eta_1, \ldots, \eta_{d'}).$$

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