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CARDINAL CONDITIONS IN NEARNESS SPACES

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In order to redress some deficiency of the topological structure and cover the topological structure and uniform structure in one structure, there have been many efforts to define new structures, notably the limit structure ([F]), the nearness structure ([H2]) and topological constructs ([H1]).

Indeed, it is known that the nearness structure is a very nice generalization of topological and uniform structures and that it is very suitable for the study of extensions of topological and uniform spaces (see [H2]). We recall that the category Top of topological spaces and continuous maps is a bicoreflective subcategory of the category Near of nearness spaces and nearness preserving maps and the category Unif of uniform spaces and uniformly continuous maps is a bireflective subcategory of Near.

The purpose of this paper is to obtain an infinite chains of bireflective and bicoreflective subcategories of Near depending on each infinite cardinal. For each infinite cardinal $\aleph$, we define $\aleph$-determined nearness spaces as those nearness space (X, ξ) such that a family $\mathcal{A}$ of subsets of X is a near family i.e., $\mathcal{A} \in \xi$ iff every subfamily $\mathcal{B}$ of $\mathcal{A}$ with $\text{card}(\mathcal{B}) < \aleph$ is a near family. Then we note that a nearness space is contigual iff it is $\aleph_0$-determined and show that the full subcategory $\aleph$-Near of Near consisting of $\aleph$-determined nearness spaces is bireflective in Near.

Moreover, we show that the full subcategory $\aleph B$-Near of Near determined by nearness spaces with bases consisting of families whose cardinals are less than $\aleph$, is bicoreflective in Near.

For the terminology, we refer to [AHS] for the category theory and to [H2] for the theory of nearness spaces.

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1. $\aleph$-Determined Nearness spaces

In this section, we are concerned with a chain of bireflective subcategories of the category Near of nearness spaces and nearness preserving maps.

NOTATION 1.1. (1) As usual, nearness spaces and nearness preserving maps will be denoted by N-spaces and N-maps, respectively.

(2) Let $\aleph$ be an infinite cardinal. Then a family of sets will be called an $\aleph$-*family* if its cardinal is less than $\aleph$. Then $\aleph_0$-families are precisely finite families.

Now using the above notations, we define a concept of $\aleph$-determined N-spaces.

DEFINITION 1.2. An N-space (X,ξ) is said to be $\aleph$-*determined* if a family $\mathcal{A}$ of subsets of X belongs to ξ whenever every $\aleph$-subfamily of $\mathcal{A}$ belongs to ξ.

The following are immediate from the definition.

REMARK 1.3. (1) It is clear that $\aleph_0$-determined N-spaces are precisely contigual spaces. Moreover, if $\aleph \leq \aleph'$, then an $\aleph$-determined N-space is clearly an $\aleph'$-determined N-space.

(2) For an N-space (X,ξ), the following are equivalent, where $\overline{\xi}$ and μ are the associated farness and covering structures of ξ, respectively.

(a) (X,ξ) is $\aleph$-determined.

(b) For any $\mathcal{A} \subseteq P(X)$, $\mathcal{A} \in \overline{\xi}$ iff there is $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} \in \overline{\xi}$ and $\mathcal{B}$ is an $\aleph$-family.

(c) For any $\mathcal{A} \subseteq P(X)$, $\mathcal{A} \in \mu$ iff there is $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} \in \mu$ and $\mathcal{B}$ is an $\aleph$-family.

(3) By (2), a uniform N-space (X,ξ) is $\aleph$-determined iff it is $\aleph$-totally bounded.

PROPOSITION 1.4. *A topological N-space (X,ξ) is $\aleph_1$-determined iff it is a Lindelöf space.*

Proof. Suppose (X,ξ) is $\aleph_1$-determined and $\mathcal{F}$ is a family of closed sets in (X,ξ) with the countable intersection property, then every countable subfamily of $\mathcal{F}$ belongs to ξ. Since (X,ξ) is $\aleph_1$-determined, $\mathcal{F}$ also belongs to ξ, so that $\mathcal{F}$ has a non-empty intersection, for (X, ξ) is topological. Thus (X, ξ) is a Lindelöf space.

For the converse, take any $\mathcal{A} \subseteq P(X)$ such that every countable subfamily of $\mathcal{A}$ belongs to ξ. Then for any countable subfamily $\mathcal{B}$ of $\mathcal{A}$, $\{\mathrm{cl}\, B : B \in \mathcal{B}\}$ has a non-empty intersection, for (X, ξ) is topological. Thus $\{\mathrm{cl}\, A : A \in \mathcal{A}\}$ is a family of closed subsets with the countable intersection property, so that it has a non-empty intersection. Thus $\mathcal{A}$ belongs to ξ. This completes the proof.

The above proposition suggests that $\aleph_1$-determined N-spaces would be called Lindelöf N-spaces.

We recall that the category Near of N-spaces and N-maps is a properly fibred topological construct (see [AHS] and [H2]). The full subcategory of Near consisting of all $\aleph$-determined N-spaces will be denoted by $\aleph$-Near.

THEOREM 1.5. *For any infinite cardinal $\aleph$, the category $\aleph$-Near is bireflective in Near.*

Proof. Take any $(X, \xi) \in$ Near and let

$$\xi^{\aleph} = \{\mathcal{A} \subseteq P(X) : \text{ for any } \aleph\text{-subfamily } \mathcal{B} \text{ of } \mathcal{A}, \mathcal{B} \in \xi\}.$$

Let us show that $\xi^{\aleph}$ is a nearness structure on X.

Suppose $\mathcal{A} < \mathcal{B}$ and $\mathcal{B} \in \xi^{\aleph}$. For any $\aleph$-subfamily $\mathcal{C}$ of $\mathcal{A}$, there is an $\aleph$-subfamily $\mathcal{D}$ of $\mathcal{B}$ with $\mathcal{C} < \mathcal{D}$. Since $\mathcal{B} \in \xi^{\aleph}$, $\mathcal{D} \in \xi$; hence $\mathcal{C} \in \xi$. Thus $\mathcal{A} \in \xi^{\aleph}$, so that $\xi^{\aleph}$ satisfies N_1). Clearly $\xi \subseteq \xi^{\aleph}$ and therefore, $\xi^{\aleph}$ satisfies N_2) and N_3). For N_4), suppose $\mathcal{A} \vee \mathcal{B} \in \xi^{\aleph}$ but $\mathcal{A} \notin \xi^{\aleph}$ and $\mathcal{B} \notin \xi^{\aleph}$, then there are $\aleph$-families $\mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{D} \subseteq \mathcal{B}$ such that $\mathcal{C} \notin \xi$ and $\mathcal{D} \notin \xi$; therefore $\mathcal{C} \vee \mathcal{D} \notin \xi$. Since $\mathcal{C} \vee \mathcal{D}$ is an $\aleph$-subfamily of $\mathcal{A} \vee \mathcal{B}$, we have a contradiction. Since $\aleph$ is an infinite cardinal, a finite family of subsets of X belongs to $\xi^{\aleph}$ iff it belongs to ξ. Thus for any $A \subseteq X$, $\mathrm{cl}_{\xi} A = \mathrm{cl}_{\xi^{\aleph}} A$. Suppose $\{\mathrm{cl}_{\xi^{\aleph}} A : A \in \mathcal{A}\} \in \xi^{\aleph}$, then for any $\aleph$-subfamily $\mathcal{B}$ of $\mathcal{A}$, $\{\mathrm{cl}_{\xi^{\aleph}} B : B \in \mathcal{B}\} = \{\mathrm{cl}_{\xi} B : B \in \mathcal{B}\} \in \xi$ and hence $\mathcal{B} \in \xi$. Thus $\mathcal{A}$ belongs to $\xi^{\aleph}$. Thus $\xi^{\aleph}$ is a nearness structure on X. Now take any family $\mathcal{A} \subseteq P(X)$ such that every $\aleph$-subfamily of $\mathcal{A}$ belongs to $\xi^{\aleph}$. Since a subfamily of an $\aleph$-family is again an $\aleph$-family, every $\aleph$-subfamily of $\mathcal{A}$ belongs to ξ so that $\mathcal{A}$ also belongs to $\xi^{\aleph}$. In all, the N-space $(X, \xi^{\aleph})$ is an object of $\aleph$-Near. Let $r : (X, \xi) \to (X, \xi^{\aleph})$ be the map given by the identity map of X, then r is an N-map for $\xi \subseteq \xi^{\aleph}$. Now take any N-map $f : (X, \xi) \to (Y, \eta)$ such that $(Y, \eta) \in \aleph$-Near and take any $\mathcal{A} \in \xi^{\aleph}$, then for any $\aleph$-family $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B} \in \xi$; hence $f(\mathcal{B}) \in \eta$. Thus $f(\mathcal{A})$ belongs to $\xi^{\aleph}$; therefore $f : (X, \xi^{\aleph}) \to (Y, \eta)$ is again an N-map. Hence $r : (X, \xi) \to (X, \xi^{\aleph})$ is the $\aleph$-Near reflection of (X, ξ), which is clearly a bimorphism.

The following are immediate from the above theorem:

COROLLARY 1.6. (1) *The category $\aleph$-Near is closed under the formations of initial sources in Near.*

(2) *The category $\aleph$-Near is a properly fibred topological construct.*

(3) *The category $\aleph$-Near is complete and cocomplete and it is closed under the formations of limits in Near. Moreover, colimits in $\aleph$-Near are precisely the reflections of the corresponding colimits in Near.*

It is well known that a topological product space of Lindelöf spaces need not be a Lindelöf space. For example, let $\mathcal{T}$ be the topology on the set R of real numbers generated by the half open intervals $[a, b)$, then $(R, \mathcal{T}) \times (R, \mathcal{T})$ is not a Lindelöf space. But let $\xi_{\mathcal{T}}$ denote the nearness structure induced by $\mathcal{T}$, then $(R, \xi_{\mathcal{T}})$ is an $\aleph_1$-determined N-space by Proposition 1.4; hence $(R, \xi_{\mathcal{T}}) \times (R, \xi_{\mathcal{T}})$ is $\aleph_1$-determined, which cannot be a topological N-space.

Since the category UNear of uniform N-spaces and N-maps, which is isomorphic with the category Unif of uniform spaces and uniformly continuous maps, is bireflective in Near and Near is a topological construct, one has the following by the above theorem.

COROLLARY 1.7. *The category UNear $\cap \aleph$-Near is also bireflective in Near and moreover, UNear $\cap \aleph$-Near is isomorphic with the category of $\aleph$-totally bounded uniform spaces and uniformly continuous maps.*

2. Bases for Nearness spaces

This section concerns with nearness spaces with bases consisting of $\aleph$-families and we then obtain an infinite sequence of coreflective subcategories of Near.

DEFINITION 2.1. Let (X,ξ) be an N-space and $\beta \subseteq \xi$. Then β is said to be a *base* for ξ of (X,ξ) if for any $\mathcal{A} \in \xi$, there is $\mathcal{B} \in \beta$ with $\mathcal{A} < \mathcal{B}$.

We note that if β is a base for an N-space (X,ξ), then we have:

$$\xi = \{\mathcal{A} \subseteq P(X) : \mathcal{A} < \mathcal{B} \text{ for some } \mathcal{B} \in \beta\}$$

and vice versa.

DEFINITION 2.2. For an infinite cardinal $\aleph$, an N-space (X,ξ) is said to be $\aleph$-*based* if it has a base consisting of $\aleph$-families.

REMARK 2.3. (1) An N-space (X,ξ) is $\aleph_0$-bases ($\aleph_1$-based, resp.) iff ξ is completely determined by the finite (countable, resp.) members of ξ. Moreover, every finite N-space is clearly $\aleph_0$-based.

(2) If $\aleph \leq \aleph'$, then an $\aleph$-based N-space is $\aleph'$-based.

(3) A topological N-space (X,ξ) is $\aleph$-based iff the following holds: for any family $\mathcal{A}$ of subsets of X with $\cap\{\mathrm{cl}\, A : A \in \mathcal{A}\} \neq \emptyset$, there is an $\aleph$-family $\mathcal{B}$ of subsets of X such that $\mathcal{A} < \mathcal{B}$ and $\cap\{\mathrm{cl}\, B : B \in \mathcal{B}\} \neq \emptyset$.

Let $\aleph B$-Near denote the full subcategory of Near determined by $\aleph$-based N-spaces.

We recall that a sink $(f_i : (X_i,\xi_i) \to (X,\xi))_{i\in I}$ is a final sink iff $\xi = \{\mathcal{A} \subseteq P(X) : \cap\mathcal{A} \neq \emptyset\} \cup \{\mathcal{A} \subseteq P(X) : f_i^{-1}(\mathcal{A}) \in \xi_i \text{ for some } i \in I\}$. Using this together with the fact that Near is a topological construct, we have the following.

THEOREM 2.4. *For any infinite cardinal $\aleph$, the category $\aleph B$-Near is a bicoreflective subcategory of Near.*

Proof. It is enough to show that $\aleph B$-Near is closed under the formations of final sinks in Near. Take any family $((X_i,\xi_i))_{i\in I}$ in $\aleph B$-Near and a final sink $(f_i : (X_i,\xi_i) \to (X,\xi))_{i\in I}$ in Near. Let $\mathcal{A}$ be an element of ξ. For the case of $\cap\mathcal{A} \neq \emptyset$, take any $x \in \cap\mathcal{A}$, then clearly one has $\mathcal{A} < \{\{x\}\}$ and the latter is an $\aleph$-family. For the case of $f_i^{-1}(\mathcal{A}) \in \xi_i$ for some $i \in I$. Since $(X_i,\xi_i) \in \aleph B$-Near, there is an $\aleph$-family $\mathcal{B}$ in ξ_i with $f_i^{-1}(\mathcal{A}) < \mathcal{B}$. Thus $f_i(\mathcal{B})$ is an $\aleph$-family of subsets of X such that $\mathcal{A} < f_i(\mathcal{B})$ and $f_i(\mathcal{B}) \in \xi$, because $f_i^{-1}(f_i(\mathcal{B}))$ corefines $\mathcal{B}$; hence $f_i^{-1}(f_i(\mathcal{B})) \in \xi_i$. Therefore, (X,ξ) is $\aleph$-based. This completes the proof.

REMARK 2.5. Let (X,ξ) be an N-space and let $\xi_{\aleph}b = \{\mathcal{A} \subseteq P(X) :$ there is an $\aleph$-family $\mathcal{B}$ in ξ with $\mathcal{A} < \mathcal{B}\}$. Then it is a routine calculation that the identity map $c : (X, \xi_{\aleph}b) \to (X,\xi)$ is the $\aleph B$-Near coreflection of (X,ξ). We left the detail to the readers.

The following is immediate from the above theorem.

COROLLARY 2.6. (1) *The category $\aleph B$-Near is closed under the formations of final sinks and colimits in Near.*

(2) *The category $\aleph B$-Near is a properly fibred topological construct.*

(3) *The category $\aleph B$-Near is complete and cocomplete. Moreover, the limits in $\aleph B$-Near is the coreflection of the corresponding limits in Near.*

(4) *A subspace of an $\aleph$-based N-space is again $\aleph$-based.*

Since the category TNear of topological N-spaces and N-maps is bicoreflective in Near, one has the following by the above theorem.

COROLLARY 2.7. *The category TNear $\cap \aleph B$-Near is a bicoreflective subcategory of Near.*

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