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SOME CHARACTERIZATIONS OF SASAKIAN MANIFOLDS WITH PARALLEL RICCI CONTRACTION OF CONTACT CONFORMAL CURVATURE TENSOR FIELD*

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1. Preliminaries

In 1949([1]), S. Bochner has introduced "Bochner curvature tensor" on a Kaehlerian manifold analogous to the Weyl conformal curvature tensor on a Riemannian manifold. However, we have not known the exact meaning of his tensor yet. In 1989, H. Kitahara, K. Matsuo and J. S. Pak([8]) defined a new tensor field on a hermitian manifold which is conformally invariant and studied some properties of this new tensor field. They called this new tensor field "conformal curvature tensor field". In particular, on a 2n-dimensional Kaehlerian manifold the conformal curvature tensor field is given by

$$\begin{aligned} B_{0,dcba} = R_{dcba} &+ \frac{1}{2n}(g_{da}R_{cb} - g_{ca}R_{db} + R_{da}g_{cb} \\ &- R_{ca}g_{db} - f_{da}S_{cb} + f_{ca}S_{db} - S_{da}f_{cb} \\ &+ S_{ca}f_{db} + 2f_{dc}S_{ba} - 2S_{dc}f_{ba}) \\ &+ \frac{(n+2)s}{4n^2(n+1)}(f_{da}f_{cb} - f_{ca}f_{db} - 2f_{dc}f_{ba} \\ &- \frac{(3n+2)s}{4n^2(n+1)}(g_{da}g_{cb} - g_{ca}g_{db}) \end{aligned}$$

where (f_c^b, g_{ab}) denotes the Kaehlerian structure, Ricci tensor and scalar curvature being respectively denoted by R_{ba} and s, and $f_{ca} = f_c{}^b g_{ba}$ and $S_{cb} = f_c{}^e R_{eb}$.

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In 1990, one of the present authors([6]) defined a new tensor field on a Sasakian manifold, which was constructed from the conformal curvature tensor field by using the Boothby-Wang's fibration and studied some properties of this new tensor field.

Particularly, we study the following facts : In section 2, we recall definitions and fundamental properties of almost contact manifold.

Section 3 is devoted to recalling contact conformal curvature tensor field on a Sasakian manifold and some properties of Sasakian manifolds concerning with vanishing contact conformal curvature tensor field. In the last section 4, we study some characterizations of Sasakian manifolds with parallel Ricci contraction of contact conformal curvature tensor field.

2. Almost contact manifold

We first of all recall definitions and fundamental properties of almost contact manifold for later use. Let M be a $(2n+1)$ - dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U; x^h\}$ in which there are given a tensor field $\phi_i{}^h$ of type (1, 1), a vector field ξ^h and 1- form η_i satisfying

$$(2.1)\qquad \begin{aligned} &\phi_j{}^h\phi_h{}^i = -\delta_j^i + \eta_j\xi^i, \phi_j{}^i\xi^j = 0,\\ &\eta_i\phi_j{}^i = 0, \quad \eta_i\xi^i = 1, \end{aligned}$$

where here and in the sequel the indices h, i, j, k, l run over the range $\{1, 2, \ldots, 2n+1\}$. Such a set (ϕ, ξ, η) of a tensor field ϕ, a vector field ξ and a 1-form η is called an almost contact structure and a manifold with an almost contact structure is called an almost contact manifold. If the Nijenhuis tensor

$$N_{ji}{}^h = \phi_j{}^k\partial_k\phi_i{}^h - \phi_i{}^k\partial_k\phi_j{}^h - (\partial_j\phi_i{}^k - \partial_i\phi_j{}^k)\phi_k{}^h$$

formed with $\phi_i{}^h$ satisfies

$$N_{ji}{}^h + (\partial_j\eta_i - \partial_i\eta_j)\xi^h = 0,$$

where $\partial_i = \partial/\partial x^i$, then the almost contact structure is said to be normal and the manifold is called a normal almost contact manifold.

Suppose that there is given, in an almost contact manifold, a Riemannian metric g_{ji} such that

$$g_{kh}\phi_i{}^k\phi_i{}^h = g_{ji} - \eta_j\eta_i, \quad \eta_i = g_{ih}\xi^h. \tag{2.2}$$

Then the almost contact structure is said to be almost contact metric structure and the manifold is called an almost contact metric manifold. In an almost contact metric manifold, the tensor field $\phi_{ji} = \phi_j{}^h g_{hi}$ is skew-symmetric.

If an almost contact metric structure satisfies

$$\phi_{ji} = \frac{1}{2}(\partial_j\eta_i - \partial_i\eta_j),$$

then the almost contact metric structure is called a contact metric structure. A manifold with a normal contact metric structure is called a Sasakian manifold. It is well known that in a Sasakian manifold we have

$$\nabla_j\xi^i = \phi_j{}^i, \tag{2.3}$$

and

$$\nabla_j\phi_i{}^h = \eta_i\delta_j^h - \xi^h g_{ji}, \tag{2.4}$$

where ∇_j denotes the operator of covariant differentiation with respect to g_{ji}. If we denote by $\mathcal{L}$ the operator of Lie derivation with respect to the vector field ξ^h, we have

$$\mathcal{L}g_{ji} = \nabla_j\eta_i + \nabla_i\eta_j = \phi_{ji} + \phi_{ij}$$

and consequently

$$\mathcal{L}g_{ji} = 0 \tag{2.5}$$

which shows that the vector field ξ^h is a Killing vector field. Now, from equation (2.3), (2.4) and the Ricci identity

$$\nabla_k\nabla_j\xi^h - \nabla_j\nabla_k\xi^h = R_{kji}{}^h\xi^i,$$

we find

$$R_{kji}{}^{h}\xi^{i} = \delta_{k}^{h}\eta_{j} - \delta_{j}^{h}\eta_{k} \tag{2.6}$$

or

$$R_{kji}{}^{h}\eta_{h} = \eta_{k}g_{ji} - \eta_{j}g_{ki}, \tag{2.7}$$

from which, by contraction,

$$R_{ji}\xi^{i} = 2n\eta_{j}. \tag{2.8}$$

From equation (2.3), (2.4) and the Ricci identity

$$\nabla_{k}\nabla_{j}\phi_{i}{}^{h} - \nabla_{j}\nabla_{k}\phi_{i}{}^{h} = R_{kji}{}^{h}\phi_{i}{}^{l} - R_{kji}{}^{l}\phi_{l}{}^{h}$$

we find

$$\begin{aligned} R_{kji}{}^{h}\phi_{i}{}^{l} - R_{kji}{}^{l}\phi_{l}{}^{h} = &-\phi_{k}{}^{h}g_{ji} + \phi_{j}{}^{h}g_{ki} \\ &- \delta_{k}^{h}\phi_{ji} + \delta_{j}^{h}\phi_{ki}, \end{aligned} \tag{2.9}$$

from which, by contraction,

$$R_{jh}\phi_{i}{}^{h} + R_{kjih}\phi^{kh} = -(2n-1)\phi_{ji}, \tag{2.10}$$

where $\phi^{kh} = \phi_{i}{}^{h}g^{ki}, g^{ki}$ being contravariant components of the metric tensor g_{ji}. Since

$$R_{kjih}\phi^{kh} = R_{hijk}\phi^{kh} = -R_{kijh}\phi^{kh},$$

we have from (2.10)

$$S_{ji} + S_{ij} = 0, \tag{2.11}$$

where $S_{ji} = \phi_{j}^{h}R_{hi}$. Moreover, the followings hold good ([7])

$$\begin{aligned} \nabla_{k}S_{j}^{k} &= \frac{1}{2}\phi_{j}{}^{k}\nabla_{k}s + (s-2n)\eta_{i}, \\ \nabla_{k}S_{ji} &= \eta_{j}R_{ik} - 2ng_{jk}\eta_{i} + \phi_{j}t\nabla_{k}R_{ti}. \end{aligned} \tag{2.12}$$

A plane section in $T_x(M)$ is called a ϕ-section if there exists a unit vector X in $T_x(M)$ orthogonal to ξ such that $\{X, \phi X\}$ is an orthonormal basis of the plane section. Then the sectional curvature $g(R(X,\phi X)\phi X, X)$ is called a ϕ-sectional curvature. If the ϕ-sectional curvature at any point of a Sasakian manifold of dimension ≥ 5 is independent of the choice of ϕ-section, then it is constant on the manifold and the curvature tensor is given by

$$\begin{aligned} R_{kjih} = {} & \frac{1}{4}(k+3)(g_{kh}g_{ji} - g_{jh}g_{ki}) + \frac{1}{4}(k-1)(\eta_k\eta_i g_{jh} \\ & - \eta_j\eta_i g_{kh} + g_{ki}\eta_j\eta_h - g_{ji}\eta_k\eta_h + \phi_{kh}\phi_{ji} \\ & - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}), \end{aligned}$$

where k is the constant ϕ-sectional curvature.

3. Contact Conformal Curvature Tensor Field

In a $(2n+1)$-dimensional Sasakian manifold M^{2n+1} is defined contact conformal curvature tensor field $C_{0,kji}{}^h$ by

(3.1)

$$\begin{aligned} C_{0,kji}{}^h = {} & R_{kji}{}^h + \frac{1}{2n}(\delta_k^h R_{ji} - \delta_j^h R_{ki} + R_k{}^h g_{ji} - R_j{}^h g_{ki} - R_k{}^h \eta_j\eta_i \\ & + R_j{}^h \eta_k\eta_i - \eta_k\xi^h R_{ji} + \eta_j\xi^h R_{ki} - \phi_k{}^h S_{ji} + \phi_j{}^h S_{ki} \\ & - S_k{}^h\phi_{ji} + S_j{}^h\phi_{ki} + 2\phi_{kj}S_i{}^h + 2S_{kj}\phi_i{}^h) \\ & + \frac{1}{2n(n+1)}[2n^2 - n - 2 + \frac{(n+2)s}{2n}](\phi_k{}^h\phi_{ji} - \phi_j{}^h\phi_{ki} \\ & - 2\phi_{kj}\phi_i{}^h) + \frac{1}{2n(n+1)}[n+2 - \frac{(3n+2)s}{2n}](\delta_k^h g_{ji} - \delta_j^h g_{ki}) \\ & - \frac{1}{2n(n+1)}[-(4n^2+5n+2) + \frac{(3n+2)s}{2n}](\delta_k^h\eta_j\eta_i - \delta_j^h\eta_k\eta_i \\ & + \eta_k\xi^h g_{ji} - \eta_j\xi^h g_{ki}), \end{aligned}$$

which is constructed from the conformal curvature tensor field (1.1) in a Kaehlerian manifold by using the Boothby-Wang's fibration ([2]), where

$s = R_{ji}g^{ji}$ denotes the scalar curvature of M^{2n+1}, $R_j{}^h = R_{ji}g^{ih}$ and $S_j{}^h = S_{ji}g^{ih}$.

In this section we assume that M^{2n+1} is a Sasakian manifold with vanishing contact conformal curvature tensor field. If $C_{0,kji}{}^h = 0$, then we have from (2.1) and (3.1)

$$0 = \frac{2(n-2)}{n}R_{ji} + \frac{1}{n}[2(n-2) - \frac{n-2}{n}s]g_{ji} + \frac{1}{n}[-2(2n+1)(n-2) + \frac{n-2}{n}s]\eta_j\eta_i$$

and consequently

$$R_{ji} = (\frac{s}{2n} - 1)g_{ji} + (2n+1-\frac{s}{2n})\eta_j\eta_i, \tag{3.2}$$

that is, M^{2n+1} is η-Einstein, provided $n \geq 2$.

Substituting (3.2) into (3.1) with $C_{0,kji}{}^h = 0$, we have

$$R_{kji}{}^h = \frac{k+3}{4}(\delta_k^h g_{ji} - \delta_j^h g_{ki}) + \frac{k-1}{4}(\phi_k{}^h\phi_{ji} - \phi_j{}^h\phi_{ki} - 2\phi_{kj}\phi_i{}^h - \delta_k^h\eta_j\eta_i + \delta_j^h\eta_k\eta_i - \eta_k\xi^h g_{ji} + \eta_j\xi^h g_{ki}) \tag{3.3}$$

where $k = \frac{1}{n(n+1)}[s - n(3n+1)]$.

Thus we have

LEMMA 3.1([6]). *A Sasakian manifold $M^{2n+1}(n \geq 2)$ with vanishing contact conformal curvature tensor field is of constant ϕ-sectional curvature $[s - n(3n+1)]/n(n+1)$.*

We suppose that the contact conformal curvature tensor coincides with the C-Bochner curvature tensor $C_{kji}{}^h$ (for the definition of $C_{kji}{}^h$, see [9]). Then it follows that

$$R_{ji} = (\frac{s}{2n} - 1)g_{ji} + (2n+1-\frac{s}{2n})\eta_j\eta_i.$$

Conversely, if M^{2n+1} is η-Einstein,

$$C_{0,kji}{}^{h} = C_{kji}{}^{h} = R_{kji}{}^{h} - \frac{k+3}{4}(\delta_k^h g_{ji} - \delta_j^h g_{ki}) - \frac{k-1}{4}(\phi_k^h \phi_{ji} - \phi_j^h \phi_{ki} - 2\phi_{kj}\phi_i^h - \delta_k^h \eta_j \eta_i + \delta_j^h \eta_k \eta_i - \eta_k \xi^h g_{ji} + \eta_j \xi^h g_{ki}).$$

Thus we have

LEMMA 3.2 ([6]). *A necessary and sufficient condition in order for* $C_{0,kji}{}^{h}$ *to coincide with* $C_{kji}{}^{h}$ *is that* M^{2n+1} *is* η*-Einstein.*

On the other hand, a direct calculation by using (2.3), (2.4) and (3.1) implies

(3.4)
$$\frac{n}{2} g^{kh}\nabla_l C_{0,kjih} = (n-2)\nabla_l R_{ji} - \frac{(n-2)}{2n}(\nabla_l s)g_{ji} + \frac{(n-2)}{2n}(\nabla_l s)\eta_j\eta_i + [-(2n+1)(n-2) + \frac{(n-2)}{2n}s](\phi_{lj}\eta_i + \eta_j\phi_{li}).$$

4. The Ricci Contraction of Contact Conformal Curvature Tensor Field

In this section, we assume that $M^{2n+1}(n > 2)$ is a Sasakian manifold with parallel Ricci contraction of contact conformal curvature tensor field, that is,

$$g^{kh}\nabla_l C_{0,kijh} = 0,$$

Then we can easily obtain from (3.4) that

$$\frac{n-2}{2}\nabla_i s - \frac{n-2}{2}\nabla_i s + \frac{n-2}{2n}(\eta^t \nabla_t s)\eta_i = 0,$$

and consequently

$$\eta^t \nabla_t s = 0, \quad \text{provided} \quad n > 2.$$

Thus we have

$$\frac{n-1}{2n}\nabla_l s = 0,$$

that is

$$s = \text{constant},$$

which together with (3.4) yields

$$\nabla_k R_{ji} = [2n+1-\frac{s}{2n}](\phi_{kj}\eta_i + \phi_{ki}\eta_j). \tag{4.1}$$

THEOREM 4.1. *A necessary and sufficient condition in order for Ricci contraction of* $C_{0,kji}{}^h$ *to be parallel is*

$$\nabla_k R_{ji} = [2n+1-\frac{s}{2n}](\phi_{kj}\eta_i + \phi_{ki}\eta_j).$$

Under the same assumptions as Theorem (4.1) it follows from (2.12) and (4.1) that

$$\nabla_k S_{ji} = \eta_j R_{ki} - 2ng_{kj}\eta_i + (2n+1-\frac{s}{2n})(g_{kj} - \eta_k\eta_j)\eta_i,$$

from which, transvecting with η^j and η^i respectively,

$$\eta^t\nabla_k S_{ti} = R_{ki} - 2n\eta_k\eta_i,$$
$$\eta^t\nabla_k S_{jt} = (1-\frac{s}{2n})(g_{kj} - \eta_k\eta_j).$$

Since

$$S_{ji} = -S_{ij}$$

the equations above imply

$$R_{kj} = (\frac{s}{2n} - 1)g_{kj} + (2n+1-\frac{s}{2n})\eta_k\eta_j \tag{4.2}$$

that is, M^{2n+1} is η-Einstein, provided $n > 2$. Thus we have

THEOREM 4.2. *If the Ricci contraction of* $C_{0,kji}{}^h$ *is parallel, then* M^{2n+1} *is* η*-Einstein.*

Combining Lemma 3.1 with Theorem 4.2, we have

COROLLARY 4.3 ([6]). *If the contact conformal curvature tensor field of a Sasakian manifold M^{2n+1} vanishes identically, then M^{2n+1} is of constant ϕ-sectional curvature.*

Combining Lemma 3.2 with Theorem 4.2, we have

COROLLARY 4.4. *If Ricci contraction of $C_{0,kji}{}^{h}$ is parallel, then the contact conformal curvature tensor field coincides with the C-Bochner curvature tensor.*

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