Comm. Korean Math. Soc. 8 (1993), No. 4, pp. 629–641

CONVERGENCE OF NONLINEAR SEMIGROUPS AND RESOLVENTS OF THEIR GENERATORS

JONG SOO JUNG AND JONG SEO PARK

1. Introduction

Let E be a real Banach space with norm $\| \ \|$. We denote the identity operator by I, and the closure of a subset D of E by $\mathrm{cl}(D)$. Let $A \subset E \times E$ be an operator with domain $D(A)$ and range $R(A)$, and $w \in \mathrm{R}$. Recall that $A + wI$ is said to be accretive (or, for short, $A \in A(w)$) if $(1 - rw)\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$ for all $r > 0, rw < 1$, and $y_i \in Ax_i$, $i = 1, 2$. A is called accretive if $w = 0$. Let $J_r = (I + rA)^{-1}, r > 0$, be the resolvent of A. If $A \in A(w)$ (A is accretive) and satisfies the range condition $R(I + rA) \supset \mathrm{cl}(D(A))$ for all $r > 0$, then $-A$ generates a semigroup S of type w (semigroup S) via the exponential formula. It was shown by Bénilan [1] that if E is a Hilbert space, then the convergence of a sequence of semigroups implies the convergence of the resolvents of their generators. This result was extended to a restricted class of Banach spaces [8]. In particular, Reich [10] provide a result in reflexive Banach spaces with a uniformly Gâteaux differentiable norm.

In this paper, we show that if E is a Banach space with a uniformly Gâteaux differentiable norm, then the convergence of a sequence of semigroups of type w implies the convergence of the resolvents of their generators. Furthermore, we investigate a condition equivalent to the convergence of the resolvents of generators of semigroups in uniformly convex and uniformly smooth Banach spaces. Our proofs are of interest in view of use of Banach limits.

Received March 8, 1993.

This study was supported by a grant of the Basic Science Research Institute, Dong-A University, 1992-1993.

2. Preliminaries

Let E be a real Banach space and let E^* be its dual. Let $U = \{x \in E : \|x\| = 1\}$ be its unit sphere of E. The norm of E is said to be Gâteaux differentiable (or E is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for each $x, y \in U$. The norm is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (1.1) is attained uniformly for $x \in U$. It is said to be uniformly Fréchet differentiable (or E is said to be uniformly smooth) if the limit is said attained uniformly for $(x, y) \in E \times E$. Every Banach space with a uniformly convex dual has a uniformly Gâteaux differentiable norm, but there are reflexive Banach spaces with a uniformly Gâteaux differentiable norm that are not even isomorphic to a uniformly smooth Banach spaces [9, P. 149]

Recall that the duality mapping F from E into the family of nonempty (by the Hahn-Banach theorem) weak-star compact convex subsets of E^* is defined by

$$F(x) = \{x^* \in E^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}$$

for each $x \in E$. It is well known that if E is smooth, then duality mapping F is single-valued. It is also known that if E has a uniformly Gâteaux differentiable norm, then F is uniformly continuous on bounded subsets of E from the strong topology of E to the weak-star topology of E^*, and that if E has a uniformly Fréchet differentiable norm, then F is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.

A Banach limit LIM is a bounded linear functional on ℓ^∞ such that

$$\inf t_n \le \text{LIM} t_n \le \sup t_n$$

and $\text{LIM} t_n = \text{LIM} t_{n+1}$ for all $\{t_n\}$ in ℓ^∞. Let $\{x_n\}$ be a bounded sequence in E. Then we can define the real valued continuous convex function ϕ on E by

$$\phi(z) = \text{LIM}\|x_n - z\|^2$$

for each $z \in E$. The following lemma was proved in [6, 7].

LEMMA 1. *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm and let $\{x_n\}$ be a bounded sequence in E. Let* LIM *be a Banach limit and $u \in C$. Then the function $\phi : E \to R$, defined by $\phi(z) = \mathrm{LIM}\|x_n - z\|^2$, attains its minimum over C at u if and only if*

$$\mathrm{LIM}(z - u, F(x_n - u)) \leq 0$$

for all $z \in C$, where F is the duality mapping of E.

Let $A \subset E \times E$ be an operator and $w \in \mathrm{R}$. Recall that A is accretive if and only if for each $x_i \in D(A)$ and $y_i \in Ax_i, i = 1, 2$, there exists $j \in F(x_1 - x_2)$ such that $(y_1 - y_2, j) \geq 0$. An operator A is said to be m-accretive if $R(I + rA) = E$ for all $r \geq 0$. If A is accretive, we can define, for each $r \geq 0$, a single-valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$. It is called the resolvent of A. The Yosida approximation of A, $A_r : R(I + rA) \to E$, is defined by $A_r = (I - J_r)/r$.

A semigroup of type w on a subset $C \subset E$ is a function $S : [0, \infty) \times C \to C$ satisfying the following conditions:

$$S(t_1 + t_2)x = S(t_1)S(t_2)x \quad \text{for } t_1,\ t_2 \geq 0 \text{ and } x \in C;$$
$$\|S(t)x - S(t)y\| \leq e^{wt}\|x - y\| \quad \text{for } t \geq 0 \text{ and } x,\ y \in C;$$
$$S(0)x = x \quad \text{for } x \in C;$$
$$S(t)x \text{ is continuous in } t \geq 0 \text{ for each } x \in C.$$

We denote by $S \in Q_w(C)$. If $w = 0$, $S(t)$ is said to be a continuous semigroup of nonlinear contractions.

Finally, we recall that if $A \in A(w)$ and the range condition $R(I + rA) \supset \mathrm{cl}(D(A))$ holds for all $r > 0$, then $-A$ generates a semigroup $S \in Q_w(\mathrm{cl}(D(A)))$ via the exponential formula [4]

3. Main results

We begin this section by proving the following lemma, which is essentially due to the work [3] of Brézis.

LEMMA 2. *Let $S \in Q_w(cl(D(A)))$ be a semigroup. Then for each $x \in cl(D(A))$, $t > 0$ and $r > 0$, we have*

$$\|x - J_r x\| \leq \frac{(r + (r + 2t)e^{wt})}{((t - r)e^{wt} + r)t} \int_0^t \|x - S(\tau)\| d\tau \tag{3.1}$$

and in particular

$$\|x - J_t x\| \le \frac{(1 + 3e^{wt})}{t} \int_0^t \|x - S(\tau)\| d\tau. \tag{3.2}$$

Proof. As usual, we denote for $x,\ y \in E$

$$\tau(x,y) = \lim_{r \downarrow 0} \frac{1}{r}(\|x + ry\| - \|x\|) = \inf_{r>0} \frac{1}{r}(\|x + ry\| - \|x\|).$$

By a result of Bénilan [2], we know that

$$\frac{1}{2}\|S(t)x - v\|^2 \le \frac{1}{2}e^{2wt}\|x - v\|^2 + \int_0^t e^{2w(t-\tau)}(v - S(s)x, y)_s ds$$

for each $v \in D(A)$ and $y \in Av$, where $(x,y)_s = \sup\{(y,x^*) : x^* \in F(x)\}$. So we have

$$\begin{aligned} \|S(t)x - v\| - e^{wt}\|v - x\| &\le \int_0^t e^{w(t-s)}\tau(v - S(s)x, y) ds \\ &\le e^{wt} \int_0^t \tau(v - S(s)x, y) ds \end{aligned}$$

for $y \in Av$. However we have for each $r > 0$

$$\tau(v - S(s)x, y) \le \frac{1}{r}(\|v - S(s)x + ry\| - \|v - S(s)x\|),$$

for $y \in Av$. If we choose $v = J_r x$, we obtain

$$\tau(J_r x - S(s)x, y) \le \frac{1}{r}(\|x - S(s)x\| - \|J_r x - S(s)x\|)$$

for $y \in AJ_r x$, and so we get

$$\|S(t)x - J_r x\| - e^{wt}\|J_r x - x\| \le \frac{e^{wt}}{r} \int_0^t (\|x - S(s)x\| - \|J_r x - S(s)x\|) ds.$$

But, $-\|J_rx - S(s)x\| \le \|x - S(s)x\| - \|x - J_rx\|$ and hence we have

$$\begin{aligned}(\frac{t}{r} - 1)e^{wt}\|J_rx - x\| &\le -\|S(t)x - J_rx\| + \frac{2}{r}e^{wt}\int_0^t \|x - S(s)x\|ds \\ &\le \|x - S(t)x\| - \|x - J_rx\| \\ &\quad + \frac{2}{r}e^{wt}\int_0^t \|x - S(s)x\|ds\end{aligned}$$

that is,

$$((\frac{t}{r} - 1)e^{wt} + 1)\|J_rx - x\| \le \|x - S(t)x\| + \frac{2}{r}e^{wt}\int_0^t \|x - S(s)x\|ds. \tag{3.3}$$

Finally, we note that

$$\|x - S(t)x\| \le \frac{1 + e^{wt}}{t}\int_0^t \|S(s)x - x\|ds. \tag{3.4}$$

Indeed

$$\begin{aligned}\|S(t)x - \frac{1}{t}\int_0^t S(s)xds\| &\le \frac{1}{t}\int_0^t \|S(t)x - S(s)x\|ds \\ &\le \frac{1}{t}\int_0^t e^{ws}\|S(t - s)x - x\|ds \le \frac{e^{wt}}{t}\int_0^t \|S(s)x - x\|ds\end{aligned}$$

and hence

$$\begin{aligned}\|x - S(t)x\| &\le \|x - \frac{1}{t}\int_0^t S(s)xds\| + \frac{1}{t}\int_0^t S(s)xds - S(t)x\| \\ &\le \frac{1}{t}\int_0^t \|x - S(s)x\|ds + \frac{e^{wt}}{t}\int_0^t \|S(s)x - x\|ds \\ &= \frac{1 + e^{wt}}{t}\int_0^t \|x - S(s)x\|ds.\end{aligned}$$

Combining (3.3) and (3.4), we obtain (3.1), and further (3.2) with $r = t$.

Now, we state and prove a main theorem.

THEOREM 1. *Let E be a Banach space with a uniformly Gâteaux differentiable norm, $\{A_n \in A(w) : n = 1, 2, ...\}$ a sequence of operators in E that satisfy the range condition, $J_r^{A_n}$ the resolvent of A_n, and $S_n \in Q_w(\mathrm{cl}(D(A_n)))$ the semigroup generated by $-A_n$. Let $A \in A(w)$ be another operator that satisfies the range condition, J_r the resolvent of A, and $S \in Q_w(\mathrm{cl}(D(A)))$ the semigroup generated by $-A$. Suppose that $S(t)x$ is uniformly continuous on bounded (t,x) sets, that $\mathrm{cl}(D(A))$ is convex, and that for each $x \in \mathrm{cl}(D(A))$, there is a sequence $\{x_n\}$ such that $x_n \in \mathrm{cl}(D(A_n))$ and $x_n \to x$. Assume that if $x_n \in \mathrm{cl}(D(A_n))$ and $x_n \to x \in \mathrm{cl}(D(A))$, then*

$$\lim_{n\to\infty} S_n(t)x_n = S(t)x \tag{I}$$

exists uniformly on bounded (t,x) sets. Then whenever $x_n \in \mathrm{cl}(D(A_n))$ and $x_n \to x \in \mathrm{cl}(D(A))$, it follows that

$$\lim_{n\to\infty} J_r^{A_n} x_n = J_r x \tag{II}$$

for each $r > 0$.

Proof. Let $x_n \to x \in \mathrm{cl}(D(A))$ and $r > 0$ with $rw < 1$, and denote $J_r^{A_n} x_n$ by y_n. Then, by (3.2) of Lemma 2, we have

$$\|x_n - y_n\| \le \frac{1 + 3e^{wr}}{r} \int_0^r \|x_n - S_n(s)x_n\| ds.$$

Since $S_n(s)x_n$ is bounded as $n \to \infty$ uniformly for $s \in [0, r]$, it follows that $\{y_n\}$ is bounded as $n \to \infty$. So for a Banach limit LIM, we can define a function $\phi : \mathrm{cl}(D(A)) \to [0, \infty)$ by

$$\phi(z) = \mathrm{LIM}\|y_n - z\|^2$$

for each $z \in \mathrm{cl}(D(A))$. Let L denote $\inf\{\phi(z) : z \in \mathrm{cl}(D(A))\}$, and consider a sequence $\{u_k\} \subset \mathrm{cl}(D(A))$ such that $\lim_{k\to\infty} \phi(u_k) = \mathrm{L}$ and $\phi(u_k) \le \phi(tu_k + (1-t)x)$ for all $0 \le t \le 1$ and all k. Then, by Lemma 1, we have

$$\mathrm{LIM}(x - u_k, F(y_n - u_k) \le 0 \tag{3.5}$$

for all k. Since $\phi(z) \to \infty$ as $\|z\| \to \infty$, the sequence $\{u_k\}$ is bounded.

For each $T > 0$ and $w_n \in \mathrm{cl}(D(A_n))$, we have

$$(3.6)\quad \frac{2}{r}\int_0^T (y_n - x_n, F(y_n - S_n(s)w_n))ds \\ \le \|y_n - w_n\|^2 - e^{-2wT}\|y_n - S_n(T)w_n\|^2$$

For each k, we choose $w_n^k \in \mathrm{cl}(D(A_n))$ such that $\lim_{n\to\infty} w_n^k = u_k$, and note that

$$\begin{aligned}
&(y_n - x, F(y_n - u_k)) - (y_n - x_n, F(y_n - S_n(s)w_n^k) \\
&= \|y_n - u_k\|^2 + (u_k - x, F(y_n - u_k)) - \|y_n - S_n(s)w_n^k\|^2 \\
&\quad - (S_n(s)w_n^k - x_n, F(y_n - S_n(s)w_n^k)) \\
&= (\|y_n - u_k\| - \|y_n - S_n(s)w_n^k\|)(\|y_n - u_k\| + \|y_n - S_n(s)w_n^k\|) \\
&\quad + (u_k - x, F(y_n - u_k) - F(y_n - S_n(s)w_n^k)) \\
&\quad - (S_n(s)w_n^k - u_k + x - x_n, F(y_n - S_n(s)w_n^k)).
\end{aligned}$$

Since $\|S_n(s)w_n^k - u_k\| \le \|S_n(s)w_n^k - S(s)u_k\| + \|S(s)u_k - u_k\|$, it follows that for each $\varepsilon > 0$ and $w \ge 0$, there are $T > 0$ and $n_0(\varepsilon)$ such that for all $0 \le s < T, n \ge n_0(\varepsilon)$ and all u_k

$$|(y_n - x, F(y_n - u_k)) - (y_n - x_n, F(y_n - S_n(s)w_n^k))| < \varepsilon,$$

$(1-e^{-2wT})/(2T/r) < rw+\varepsilon$, and $\phi(u_k) \le L+(2T/r)\varepsilon$ for all $k \ge n_0(\varepsilon)$. Consequently, we have,

$$\begin{aligned}
&\frac{2T}{r}(y_n - x, F(y_n - u_k)) \\
&\le \frac{2}{r}\int_0^T (y_n - x_n, F(y_n - S_n(s)w_n^k))ds + \frac{2T}{r}\varepsilon \\
&\le \|y_n - w_n^k\|^2 - e^{-2wT}\|y_n - S_n(T)w_n^k\|^2 + \frac{2T}{r}\varepsilon \\
&\le (\|y_n - u_k\| + \|u_k - w_n^k\|)^2 - e^{-2wT}\|y_n - S_n(T)w_n^k\|^2 + \frac{2T}{r}\varepsilon.
\end{aligned}$$

for $n \geq n_0(\varepsilon)$, such a $T > 0$, and all u_k, and hence

$$\frac{2T}{r}\mathrm{LIM}(y_n - x, F(y_n - u_k)) \leq \phi(u_k) - e^{-2wT}\phi(S(T)u_k) + \frac{2T}{r}\varepsilon$$

for all u_k. So if n is large enough and T is small enough, we have

$$\begin{aligned}(3.7) \qquad \mathrm{LIM}(y_n - x, F(y_n - u_k)) &\leq \frac{(1 - e^{-2wT})r}{2T}\mathrm{L} + 2\varepsilon \\ &< rw\mathrm{L} + (2 + \mathrm{L})\varepsilon\end{aligned}$$

for all $k \geq n_0(\varepsilon)$. Combining (3.5) and (3.7), we have

$$\mathrm{L} \leq \phi(u_k) \leq rw\mathrm{L} + (2 + \mathrm{L})\varepsilon$$

for all $k \geq n_0(\varepsilon)$. It follows that $\mathrm{L} = 0$ and that $\{u_k\}$ is a Cauchy sequence. For $w < 0$, by setting $w = -w'$ with $w' > 0$, we have the same result. Thus $\phi(u) = 0$ for some $u \in \mathrm{cl}(D(A))$ and there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that the strong $\lim y_{n_k} = u$.

For $\lambda > 0$, let $z_\lambda = (I + (r/\lambda)(I - S(\lambda)))^{-1}x$. We know that the strong $\lim_{\lambda\to 0} z_\lambda = J_r x = v$(cf. [8]). We will now show that $u = v$. Put $w_n \to z_\lambda$ in (3.6) and $n = n_k$ and let $n_k \to \infty$. Then we have, for $w \geq 0$,

$$(3.8) \quad \frac{2}{r}\int_0^T (u - x, F(u - S(s)z_\lambda))ds \leq \|u - z_\lambda\|^2 - e^{-2wT}\|u - S(T)z_\lambda\|^2.$$

It follows that given $\varepsilon > 0$,

$$\frac{2T}{r}(u - x, F(u - v)) \leq \|u - z_\lambda\|^2 - e^{-2wT}\|u - S(T)z_\lambda\|^2 + \frac{2T}{r}\varepsilon$$

if T and λ are small enough. Choosing $\lambda = T$, we obtain

$$\begin{aligned}&\frac{2\lambda}{r}(u - x, F(u - v)) \\ &\leq \|u - z_\lambda\|^2 - e^{-2w\lambda}\|u - z_\lambda - \frac{\lambda}{r}(z_\lambda - x)\|^2 + \frac{2\lambda}{r}\varepsilon \\ &\leq \|u - z_\lambda\|^2 - e^{-2w\lambda}\|u - z_\lambda\|^2 + e^{-2w\lambda}\frac{2\lambda}{r}(z_\lambda - x, F(u - z_\lambda)) + \frac{2\lambda}{r}\varepsilon\end{aligned}$$

so that

$$(u-x,F(u-v))-e^{-2w\lambda}(z_\lambda-x,F(u-z_\lambda)) \le \frac{r(1-e^{-2w\lambda})}{2\lambda}\|u-z_\lambda\|^2+\varepsilon.$$

Letting $\lambda \to 0$, we get $\|u-v\|^2(1-rw) \le 0$, and hence $\|u-v\| \le 0$. For $w < 0$, we have the same result. Thus $u = v$.

Finally, suppose that a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ converges strongly to y. Then we get just the same inequality as in (3.8) with y replaced by u. Therefore, by the same method, we obtain $y = v$ and hence the whole sequence $\{y_n\}$ converges strongly to $J_r x$.

Theorem 1 is a partial improvement of [10, Theorem 1].

REMARK 1. It is known [5] that (II) implies (I) in any Banach space (even if $\mathrm{cl}(D(A))$ is not convex). Consequently, (I) and (II) are equivalent in Theorem 1. In contrast with the linear case, (I) does not imply (II) in all Banach spaces [4].

REMARK 2. It $\mathrm{cl}(D(A)) \subset \mathrm{cl}(D(A_n))$ for all n, then (I) is implied by the assumption that $\lim_{n\to\infty} S_n(t)x = S(t)x$ uniformly on bounded (t,x) sets ($x \in \mathrm{cl}(D(A))$). If the norm of E^* is Fréchet differentiable and A is $m-$accretive, then $\mathrm{cl}(D(A))$ is convex [9, P. 160].

COROLLARY 1. *Let E be a Banach space. Assume that the norm of E is uniformly Gâteaux differentiable norm and the norm of its E^* is Fréchet differentiable. Let $\{A_n : n = 1,2,...\}$ be a sequence of $m-$accretive in E, $J_r^{A_n}$ the resolvent of A_n, and S_n the semigroup generated by $-A_n$. Let A be another $m-$accretive operator, J_r the resolvent of A, and S the semigroup generated by $-A$. Suppose that $S(t)x$ is uniformly continuous on bounded (t,x) sets, and that $\mathrm{cl}(D(A)) \subset \mathrm{cl}(D(A_n))$ for all n. If*

$$\lim_{n\to\infty} S_n(t)x = S(t)x$$

exists uniformly on bounded (t,x) sets ($x \in \mathrm{cl}(D(A))$), then

$$\text{(III)} \qquad \lim_{n\to\infty} J_r^{A_n}x = J_r x$$

exists for each $x \in \mathrm{cl}(D(A))$ and $r > 0$.

REMARK 3. If $\{A_n\}$ and A are m-accretive, then (II) is equivalent to (III).

4. Equivalent conditions

Denote the element of minimal norm in Ax by $A^o x$, and consider the following condition:

$$\text{(IV)} \qquad \begin{array}{l} \text{For each } x \in D(A), \text{there is a sequence } x_n \in D(A_n) \\ \text{such that } x_n \to x \text{ and } A_n^o x_n \to A^o x. \end{array}$$

THEOREM 2. *Let E be a Banach space that is both uniformly convex and uniformly smooth, and let $\{A_n : n = 1, 2, ...\}$ and A be m-accretive. If $D(A)$ is closed, then the property* (III) *is equivalent to* (IV).

Proof. (III) $\to$ (IV). In order to show that (III) $\to$ (IV), we use a variant of the idea of [3]. Let $x \in D(A)$. Given $\varepsilon > 0$, there is a $r > 0$ such that

$$\|x - J_r x\| < \frac{\varepsilon}{2} \text{ and } \|A^o x - A_r x\| < \frac{\varepsilon}{2}$$

Next, by (III), there is an integer N such that for $n \geq N$

$$\|J_r^{A_n} x - J_r x\| < \frac{\varepsilon}{2} \text{ and } \|(A_n)_r x - A_r x\| < \frac{\varepsilon}{2}.$$

Combining these estimates, we see that given $\varepsilon > 0$, there are an integer $N(\varepsilon)$, sequences $u_n(\varepsilon) = J_r^{A_n} x$ and $f_n(\varepsilon) = (A_n)_r x$ such that $f_n(\varepsilon) \in Au_n(\varepsilon)$ and for $n \geq N(\varepsilon)$,

$$\|u_n(\varepsilon) - x\| < \varepsilon \text{ and } \|f_n(\varepsilon) - A^o x\| < \varepsilon.$$

Let $N_k = N(\frac{1}{k})$. Then we can always assume that n_k is increasing to ∞.

Now we define the sequences x_n and g_n by $x_n = u_n(\frac{1}{k})$ and $g_n = f_n(\frac{1}{k})$ for $N_k \leq n < N_{k+1}$. Then $g_n \in A_n x_n$ and for $N_k \leq n < N_{k+1}$, we have

$$\|x_n - x\| < \frac{1}{k} \text{ and } \|g_n - A^o x\| < \frac{1}{k}.$$

Consequently, $x_n \to x$ and $g_n \to A^o x$. Now we will prove that $A_n^o x_n \to A^o x$. Indeed, $\|A_n^o x_n\| \leq \|g_n\|$ and hence there exists a subsequence

$\{A^o_{n_j}x_{n_j}\}$ of $\{A^o_n x_n\}$ that converges weakly to y for some $y \in E$. Let $v \in D(A)$. Then we have

$$((A_n)_r v - A^o_n x_n, F(J^{A_n}_r v - x_n)) \geq 0.$$

Since F is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*, we obtain

$$(A_r v - y, F(J_r v - x)) \geq 0$$

as $n_j \to \infty$. Next, letting $r \to 0$, we have

$$(A^o v - y, F(v - x)) \geq 0$$

for each $v \in D(A)$. Therefore $y \in Ax$. Since, on the other hand, $\|y\| \leq \|A^o x\|$, we have $y = A^o x$. By the uniqueness of the limit and the fact $\limsup \|A^o_n x_n\| \leq \|A^o x\|$, we conclude that $A^o_n x_n \to A^o x$.

(IV) $\to$ (III). Let $x \in \mathrm{cl}(D(A))$, $r > 0$ and $y_n = J^{A_n}_r x$. If $z \in D(A)$, $z_n \to z$ and $A^o_n z_n \to A^o z$, then $z_n = J^{A_n}_r(z_n + rA^o_n z_n)$ and $\|y_n - z_n\| \leq \|x - z_n - rA^o_n z_n\|$, so that $\{y_n\}$ is bounded. Let C be a closed convex subset of $\mathrm{cl}(D(A))$ that contains x and is invariant under J_λ for all $\lambda > 0$, and let LIM be a Banach limit. Then we can define a function ϕ on C by

$$\phi(z) = \mathrm{LIM}\|y_n - z\|^2$$

for each $z \in C$. Since ϕ is continuous, convex and $\phi(z) \to \infty$ as $\|z\| \to \infty$, while E is reflexive, ϕ attains its minimum over C at $u \in C$. So by Lemma 1, we have

$$\mathrm{LIM}(z - u, F(y_n - u)) \leq 0$$

for all $z \in C$. Since $(x - y_n - rA^o_n z_n, F(y_n - z_n)) \geq 0$, we also have for $z \in D(A)$,

$$\mathrm{LIM}(y_n - x, F(y_n - z)) \leq r\mathrm{LIM}(-A^o z, F(y_n - z)).$$

Given $\varepsilon > 0$, choose $\lambda > 0$ such that $|(A^o u - A_\lambda u, F(y_n - u))| < \varepsilon$ for all n. We obtain

$$\mathrm{LIM}(y_n - x, F(y_n - u)) \leq (\frac{r}{\lambda})\mathrm{LIM}(J_\lambda u - u, F(y_n - u)) + r\varepsilon \leq r\varepsilon.$$

But we also have $\mathrm{LIM}(x-u, F(y_n-u)) \le 0$, so that

$$\mathrm{LIM}\|y_n-u\|^2 \le r\varepsilon.$$

Thus we can choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $n_k \to \infty$ and $u = \lim y_{n_k}$. Since $(x-u-rA^o z, F(u-z)) \ge 0$ for all $z \in D(A)$, we have $u = J_r x$. Finally, suppose that a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ converges strongly to v. Then we also have

$$(x-v-rA^o z, F(v-z)) \ge 0$$

for all $z \in D(A)$ and hence $v = J_r x$. Therefore we obtain $u = v$ and the whole sequence $\{y_n\}$ converges strongly to $J_r x$.

Note that the proofs of Theorem 2 are simpler than [10, P. 80] on account of using Banach limit.

REMARK 4. It follows that in case of Theorem 2, (I), (II), (III), and (IV) are all equivalent. The Hilbert space case is due to Brézis [3].

References

1. Ph. Bénilan, *Une remarque sur la convergence des semigroupes nonlinéaires*, C. R. Acad. Sci. Paris **272** (1971), 1182–1184.
2. ———, *Equations d'évolution dans un espace de Banach quelconque et application*, Thése, Orsay, 1972.
3. H. Brézis, *New results concerning monotone operators and nonlinear semigroups* in Proceedings of a Symposium on Functional and Numerical Analysis of Nonlinear Problems, 1–26, Kyoto, 1974.
4. M. G. Crandall and T. M. Liggett.*Generation of semigroups of nonlinear transformations on general Banach spaces*, Amer. J. Math. **93** (1973), 265–298.
5. J. A. Goldstein, *Approximation of nonlinear semigroups and evolution equations*, J. Math. Soc. Japan **24** (1972), 558–573.
6. K. S. Ha and J. S. Jung, *On generators and nonlinear semigroups in Banach spaces*, J. Korean Math. Soc. **25** (1988), 245–257.
7. ———, *Strong convergence theorems for accretive operators in Banach spaces*, J. Math. Anal. Appl. **147** (1990), 330–339.
8. Y. Kobayashi, *A remark on convergence of nonlinear semigroups*, Proc. Japan Acad. **55** (1979), 45–48.
9. S. Reich, *Product formula, nonlinear semigroup, and accretive operators*, J. Funct. Anal. **36** (1980), 147–168.

10. ______, *Convergence and approximation of nonlinear semigroups*, J. Math. Anal. Appl. **76** (1980), 77–85.

Department of Mathematics
Dong-A University
Pusan 604-714, Korea

Department of Mathematics
Graduate School
Dong-A University
Pusan 604-714, Korea